28) Bose-Einstein-Kondensation in d Dimensionen

a) Zustandsdichte:

$$\sum_p \to \frac{V}{h^d}\int \underbrace{d^d p}_{dp\, p^{d-1}\, d\Omega_d} = \frac{V}{h^d} S_d \int \underbrace{dp\, p^{d-1}} = \int dE\, g(E)$$

$$p = \left(\frac{E}{a}\right)^{1/r} \Rightarrow dp = \frac{1}{a^{1/r}}\frac{1}{r}E^{\frac{1}{r}-1}dE$$

$$p^{d-1} = \left(E/a\right)^{\frac{d-1}{r}}$$

$$dp\, p^{d-1} = dE\, \frac{E^{\frac{d}{r}-1}}{r a^{d/r}}$$

$$\Rightarrow \boxed{g(E) = \underbrace{\frac{V}{h^d}\frac{S_d}{r a^{d/r}}}_{=:g_0} E^{d/r-1}}$$

mit $S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$, vgl. Bsp. 12.

b) Grosskanon. Potential:

$$Z_{GK} = \prod_p \frac{1}{1-e^{-\beta(E_p-\mu)}} \Rightarrow \beta\Omega = -\ln Z_{GK} = \sum_p \ln\left(1-e^{-\beta(E_p-\mu)}\right)$$

$$\cong \underbrace{\ln(1-z)}_{\sim \text{Grundzustand}} + \underbrace{\int dE\, g(E)\ln\left(1-z e^{-\beta E}\right)}$$

2. Term:

$$g_0 \int dE\, E^{d/r-1}\ln(1-ze^{-\beta E})$$

$$\overset{\text{p.I.}}{=} g_0 \frac{r}{d}\underbrace{E^{d/r}\ln(1-ze^{-\beta E})\Big|_0^\infty}_{0} - g_0\frac{r}{d}\int_0^\infty dE\, E^{d/r}\frac{\beta z e^{-\beta E}}{1-ze^{-\beta E}}$$

$$= -g_0\beta\frac{r}{d}\int_0^\infty dE\, E^{d/r}\frac{1}{z^{-1}e^{\beta E}-1} = -g_0\frac{1}{\beta^{d/r}}\frac{r}{d}\underbrace{\int_0^\infty dx\, \frac{x^{d/r}}{z^{-1}e^x-1}}_{\substack{\Gamma(d/r+1)\, g_{d/r+1}(z) \\ d/r\, \Gamma(d/r)}}$$

$$\Rightarrow \boxed{\beta\Omega = \ln(1-z) - g_0\frac{\Gamma(d/r)}{\beta^{d/r}} g_{d/r+1}(z)}$$

Innere Energie:

$$U = \sum_p \frac{E_p}{e^{\beta(E_p-\mu)}-1} = \frac{\partial(\beta\Omega)}{\partial\beta}\Big|_{z,V} = g_0\frac{d}{r}\frac{\Gamma(d/r)}{\beta^{d/r+1}}g_{d/r+1}(z)$$

$$= \frac{d}{r\beta}\Big(\ln(1-z) - \underbrace{\beta\Omega}_{-pV}\Big) = \frac{d}{r}\left(pV + \frac{1}{\beta}\ln(1-z)\right) \xrightarrow[\substack{\text{da } 1-z\sim\frac{1}{V}, \\ \text{siehe unten.}}]{V\to\infty} \frac{d}{r}pV$$

Bsp: $d=3$ Dimensionen:

- nichtrelativistisch ($r=2$): $U = \frac{3}{2}pV$
- ultrarelativistisch ($r=1$): $U = 3pV$

$\to$ vgl. Bspe 26, 27 für Fermionen

c) Mittlere Teilchendichte:

$$\langle N\rangle = \sum_p \frac{1}{e^{\beta(E_p-\mu)}-1} = -\frac{\partial\Omega}{\partial\mu} = -\frac{\partial\Omega}{\partial z}\frac{\partial z}{\partial \mu} = -\beta z \frac{\partial \Omega}{\partial z} =$$

$$= -\beta z\left[-\frac{1}{\beta}\frac{1}{1-z} - g_0 \frac{r}{d}\frac{\Gamma(d/r+1)}{\beta^{d/r+1}} \underbrace{\frac{d}{dz} g_{d/r+1}(z)}\right]$$

aus $g_\alpha(z) = \sum_{n=1}^{\infty}\frac{z^n}{n^\alpha}$: $z\frac{d}{dz} g_\alpha(z) = g_{\alpha-1}(z)$

$$= \frac{z}{1-z} + g_0\frac{r}{d}\frac{\Gamma(d/r+1)}{\beta^{d/r}} g_{d/r}(z)$$

$$= \frac{z}{1-z} + g_0\frac{\Gamma(d/r)}{\beta^{d/r}} g_{d/r}(z)$$

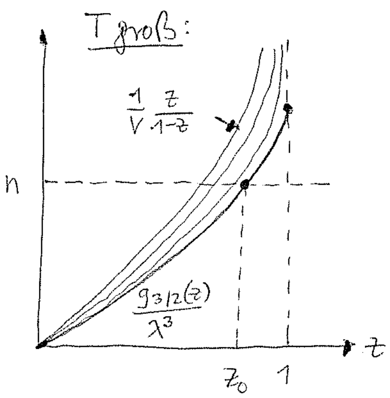
$$\Rightarrow n = \frac{\langle N\rangle}{V} = \underbrace{\frac{1}{V}\frac{z}{1-z}}_{\sim \frac{1}{V}} + \underbrace{\frac{S_d}{rh^d}\frac{\Gamma(d/r)}{(a\beta)^{d/r}}}_{\text{hängt von Temperatur ab.}} g_{d/r}(z)$$

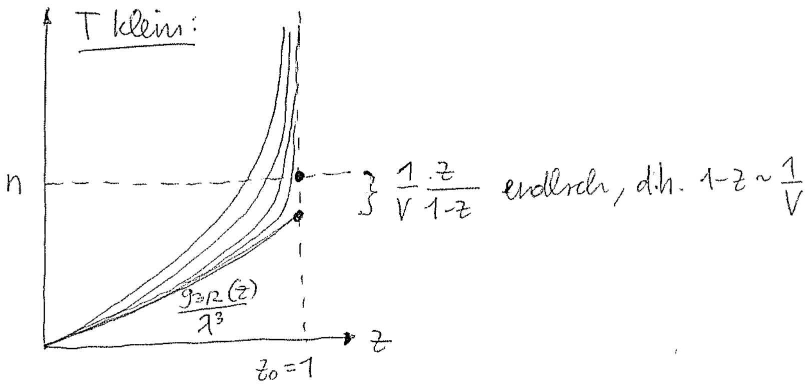
... für n, β = const: Bestimmungsgleichung für z, $z \in [0,1]$

Für $r=2$, $a=\frac{1}{2m}$ $\Rightarrow \frac{1}{\lambda^d}$

Uns interessiert $V \to \infty$. z.B. für $d=3$, $r=2$, $a=\frac{1}{2m}$ halten wir (siehe Vorlesung):

$$n = \frac{1}{V}\frac{z}{1-z} + \frac{g_{3/2}(z)}{\lambda^3}$$





D.h. für $T > T_c$ $\Rightarrow z < 1$, der Grundzustandsterm verschwindet für $V\to\infty$

$T < T_c$ $\Rightarrow z = 1$, $\frac{1}{V}\frac{z}{1-z} = \frac{N_0}{V}$ bleibt endlich.

Kritische Bedingung: $n = \frac{g_{3/2}(1)}{\lambda^3} = \frac{\zeta(3/2)}{\lambda^3}$.

Was passiert im allg. Fall? Untersuche $g_{d/r}(1) = \zeta(d/r)$:

$$g_\alpha(1) = \frac{1}{\Gamma(\alpha)}\int_0^\infty dx\, \frac{x^{\alpha-1}}{e^x-1}$$

$\Rightarrow$ keine Probleme für $x\to\infty$, aber für $x \to 0$: $e^x \approx 1 + x + \dots$

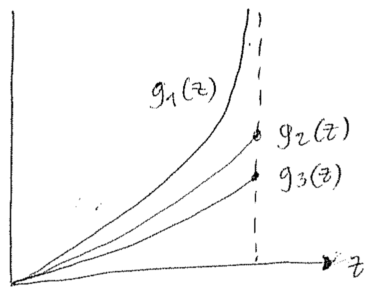
$\Rightarrow$ Integrand: $x^{\alpha-2}$

$\Rightarrow$ Integral konvergiert nur für $\alpha - 2 > -1$, dh $\alpha > 1$

$\Rightarrow \zeta(\alpha \le 1)$ divergiert.

$\Rightarrow$ dann hat n immer Schnittpunkt für $z < 1$, keine BEC.

D.h. $d/r > 1$, damit Bose-Einstein-Kondensation eintreten kann. Ansonsten divergiert $g_{d/r}(1)$, und $n < \dots g_{d/r}(1) = \infty$ ist immer erfüllt.



$d=3,\ r=2 \rightarrow$ BEC (nichtrelativistisch)
$d=3,\ r=1 \rightarrow$ BEC (ultrarelativistisch)

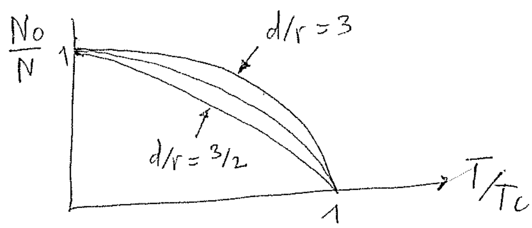
$d=2,\ r=2 \rightarrow$ keine BEC! (nichtrel.)
$d=2,\ r=1 \rightarrow$ BEC

d) $N_0(T) = ?$

$$\frac{N}{V} = \frac{N_0}{V} + (\dots)\, g_{d/r}(z) \ ; \quad \text{im Kondensationsgebiet} : z = 1$$

$$\Rightarrow \frac{N_0}{N} = 1 - \frac{S_d\, \Gamma(d/r)\, \zeta(d/r)}{n r h^d a^{d/r}} (kT)^{d/r} = 1 - \left(\frac{kT}{kT_c}\right)^{d/r}$$

$$\Rightarrow kT_c = \left[\frac{n r h^d a^{d/r}}{S_d\, \Gamma(d/r)\, \zeta(d/r)}\right]^{r/d}$$



29) Weisser Zwerg

a) $^4He \rightarrow \underbrace{2p, 2n}_{\text{Masse } m_p},\ \underbrace{2e}_{m}$ $\Rightarrow$ Insgesamt N Elektronen, N Protonen, N Neutronen:

$$M = (m + 2m_p)N \approx 2 m_p N$$

$$V = \frac{4\pi}{3} R^3$$

$$\Rightarrow n = \frac{N}{V} = \frac{3M}{8\pi m_p R^3}$$

Impulskugel:

$$N = \underset{\uparrow \text{Spin}}{2} \frac{V}{h^3} \int^{p_F} d^3p = \frac{2V}{(2\pi\hbar)^3} \frac{4\pi}{3} p_F^3 = \frac{V}{3\pi^2\hbar^3} p_F^3 \quad \Rightarrow \quad p_F = \hbar\,(3\pi^2 n)^{1/3}$$

$$\Rightarrow x_F = \frac{p_F}{mc} = \frac{\hbar}{mc}(3\pi^2 n)^{1/3} = \frac{\hbar}{mc}\left(\frac{9\pi M}{8 m_p R^3}\right)^{1/3} = \frac{\hbar}{mc}\left(\frac{9\pi}{8 m_p}\right)^{1/3} \frac{M^{1/3}}{R}$$

und:

$$\sum_p \rightarrow \frac{V}{\pi^2\hbar^3} \int dp\, p^2$$

b) Energie:

1 Teilchen: $E_p = \sqrt{m^2c^4 + p^2c^2}$

Gesamt: $E_c = \int dE\, g(E) \underbrace{f(E)}_{\approx 1,\ \text{da } T \approx 0} E$

bzw:

$$E_c = \sum_p E_p \simeq \frac{V}{\pi^2\hbar^3} \int_0^{p_F} dp\, p^2 \sqrt{m^2c^4 + p^2c^2} = \frac{V}{\pi^2} \frac{mc^2}{\hbar^3} \int_0^{p_F} dp\, p^2 \sqrt{1 + \underbrace{\frac{p^2}{m^2c^2}}_{x^2}}$$

$$= \frac{V}{\pi^2} \left(\frac{mc}{\hbar}\right)^3 mc^2 \int_0^{x_F} dx\, x^2 \sqrt{1+x^2}$$

c) Nichtrelativistisch:

$$x_F \ll 1 \Rightarrow \int^{x_F} dx\, x^2 \sqrt{1+x^2} \approx \int dx\, x^2 \left(1 + \frac{x^2}{2} + ..\right) = \frac{x_F^3}{3} + \frac{x_F^5}{10} + \ldots$$

$$\frac{x_F^3}{3} = \left(\frac{\hbar}{mc}\right)^3 \frac{(3\pi^2 n)}{3} \Rightarrow E_c = N mc^2 \checkmark$$

Ultrarelativistisch:

$$x_F \gg 1 \Rightarrow \int dx\, x^2 \sqrt{1+x^2} = \int_0^{x_F} dx\, x^3 \sqrt{1 + \frac{1}{x^2}} \approx \int_0^{x_F} dx\, x^3 \left(1 + \frac{1}{2x^2} + ..\right) = \frac{x_F^4}{4} + \frac{x_F^2}{4} + ..$$

Integrand für kleine x klein

$\sim R^3$

$$\Rightarrow E_c = \frac{V}{4\pi^2} \left(\frac{mc}{\hbar}\right)^3 mc^2 x_F^4 \left(1 + \frac{1}{x_F^2}\right)$$

$$= \underbrace{\frac{\hbar c}{3\pi} \left(\frac{9\pi M}{8 m_p}\right)^{4/3}}_{c_1} \frac{1}{R} \left[1 + \underbrace{\left(\frac{mc}{\hbar}\right)^2 \left(\frac{8 m_p}{9\pi M}\right)^{2/3}}_{c_2} R^2 + \ldots\right]$$

d) Nullpunktsdruck, $x_F \gg 1$:

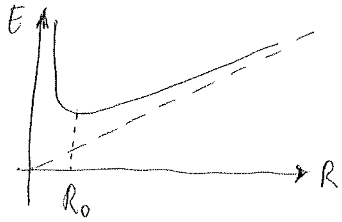
$$P = -\frac{\partial E_c}{\partial V} = -\frac{1}{4\pi^2} \left(\frac{mc}{\hbar}\right)^3 mc^2 \left[x_F^4 + \underbrace{V \frac{\partial}{\partial V} x_F^4}\right]$$

$$V \cdot 4 x_F^3 \underbrace{\frac{\partial x_F}{\partial n}}_{\frac{x_F}{3n}} \underbrace{\frac{\partial n}{\partial V}}_{-\frac{N}{V^2} = -\frac{n}{V}} = -\frac{4}{3} x_F^4$$

$$= \frac{1}{12\pi^2} \left(\frac{mc}{\hbar}\right)^3 mc^2 x_F^4$$

$$= \frac{\hbar c}{12\pi^2} (3\pi^2 n)^{4/3}$$

e) $E = \underbrace{\frac{c_1}{R} + c_1 c_2 R}_{\text{Elektronen}} \underbrace{- \frac{3}{5}\frac{GM^2}{R}}_{\text{He-Kerne}} = \underbrace{\left(c_1 - \frac{3}{5}GM^2\right)}_{>0}\frac{1}{R} + c_1 c_2 R$



d.h. Minimum nur für

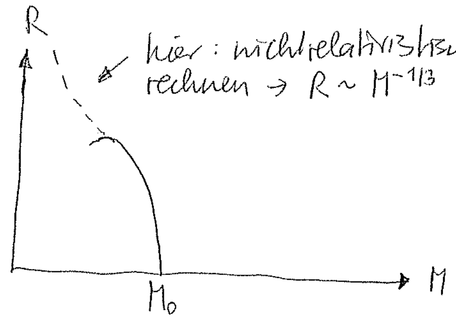
$$c_1 > \frac{3}{5}GM^2 \Rightarrow \frac{5}{3G}\frac{\hbar c}{3\pi}\left(\frac{9\pi}{8m_p}\right)^{4/3} > M^{2/3}$$

$$\Rightarrow M < M_0 = \left[\frac{\hbar c}{3\pi}\left(\frac{9\pi}{8m_p}\right)^{4/3}\frac{5}{3G}\right]^{3/2} = 1.73\, M_\odot \quad , \text{ mit } M_\odot \approx 2 \cdot 10^{30}\, kg$$

$\underline{\underline{R_0(M):}}$ $$\frac{\partial E}{\partial R} = -\frac{1}{R^2}\left(c_1 - \frac{3}{5}GM^2\right) + c_1 c_2 = 0$$

$$\Rightarrow R^2 = \frac{1}{c_2}\left(1 - \frac{3GM^2}{5c_1}\right) = \frac{1}{c_2}\left(1 - \left(\frac{M}{M_0}\right)^{2/3}\right)$$

$$\Rightarrow R = \frac{\hbar}{mc}\left(\frac{9\pi}{8m_p}\right)^{1/3} M^{1/3}\sqrt{1 - \left(\frac{M}{M_0}\right)^{2/3}}$$



By the way: typische Grössenordungen:

$x_F \sim 2$

$E_F = mc^2\sqrt{1 + x_F^2} \sim 1\, MeV$

$T = 10^7 K \Rightarrow kT \sim 1\, keV \ll E_F$

$M \sim M_\odot$

$R \sim 5000\, km$