Blatt 12

32) a) $g = g_0 + g_E$

$$g_E = 3N\,\delta(\omega - \omega_E)$$

b) $E = \sum_{k,\lambda} E_{k,\lambda} = \sum_{k,\lambda} \hbar\omega_{k\lambda}\left(\langle n\rangle + \frac{1}{2}\right)$

$$\langle n\rangle = \frac{1}{\exp\left(\frac{\hbar\omega_E}{k_BT}\right)-1} \quad , \; E_E = 3N\hbar\omega_E\left(\frac{1}{\exp(..)-1} + \frac{1}{2}\right)$$

$$C_V = \left(\frac{\partial E}{\partial T}\right)_V = \frac{3N}{k_BT^2}\,\frac{(\hbar\omega_E)^2}{\left(\exp\left(\frac{\hbar\omega_E}{k_BT}\right)-1\right)^2}\exp(..) \qquad \Theta_E = \frac{\hbar\omega_E}{k_B}$$

$$= 3Nk_B\left(\frac{\Theta_E}{T}\right)^2 \underbrace{\frac{\exp\left(\frac{\Theta_E}{T}\right)}{\left[\exp\left(\frac{\Theta_E}{T}\right)-1\right]^2}}_{\left(\frac{\Theta_E}{T}\right)^2 \frac{1}{\left(\Theta_E/T\right)^2} = 1}$$

$T \to \infty$

$$= 3Nk_B$$

$$E = E_D + E_E$$

$$C_V = C_{VD} + C_{VE}$$

$$C_V = 3Nk_B + 3Nk_B$$

$$= 2\cdot 3Nk_B = 6Nk_B$$

$$T \to 0 \;, \; C_V \sim T^3$$

33) a) $Z = \sum_{S_1} \dots \sum_{S_N} e^{-\beta H}$, $H = -\frac{J}{2N} \sum_{i,j} S_i S_j - B \sum_i S_i$

$S_i \in \{-1, 1\}$

$$\sum_{i,j}^{N} S_i S_j = \sum_i S_i \sum_j S_j = \left(\sum_i S_i\right)^2$$

HST: $$e^{\frac{\beta J}{2N}(\sum S)^2} = \frac{1}{\sqrt{\pi}} \int dx \, e^{-x^2 + 2x\sqrt{\frac{\beta J}{2N}} \sum S_i}$$

$$e^{\frac{\beta J}{2N}(\sum S)^2 + \beta B \sum_i S_i} = \frac{1}{\sqrt{\pi}} \int dx \, e^{-x^2 + (\sum S_i)(\beta B + x\sqrt{\frac{2\beta J}{N}})}$$

$x = \sqrt{\frac{N\beta J}{2}}\, y$ $$= \sqrt{\frac{N\beta J}{2\pi}} \int dy \, e^{-\frac{N\beta J}{2} y^2 + \beta(Jy+B)\sum_i S_i}$$

$$Z = \sqrt{\frac{N\beta J}{2\pi}} \int dy \, e^{-\frac{N\beta J}{2} y^2} \underbrace{\sum_{\{S_i\}} \prod_i e^{\beta(Jy+B)S_i}}_{=*}$$

[NR: $\sum_{S_1} \dots \sum_{S_N} \prod_i A^{S} = \sum_{S_1} A^{S_1} \dots \sum_{S_N} A^{S_N} = \left(\sum_S A^S\right)^N$]

$$* = \left(\sum_{\substack{S=+1\\S=-1}} e^{\beta(Jy+B)S}\right)^N$$

$$= 2^N \cosh^N(\beta(Jy+B))$$

$$\mathcal{S}(y, B, \beta) = \frac{\beta J y^2}{2} - \ln\left[2\cosh\{\beta(Jy+B)\}\right]$$

$$Z = \sqrt{\frac{N\beta J}{2\pi}} \int dy \, e^{-N\mathcal{S}(y, B, \beta)}$$

b) <u>Sattelp.</u>

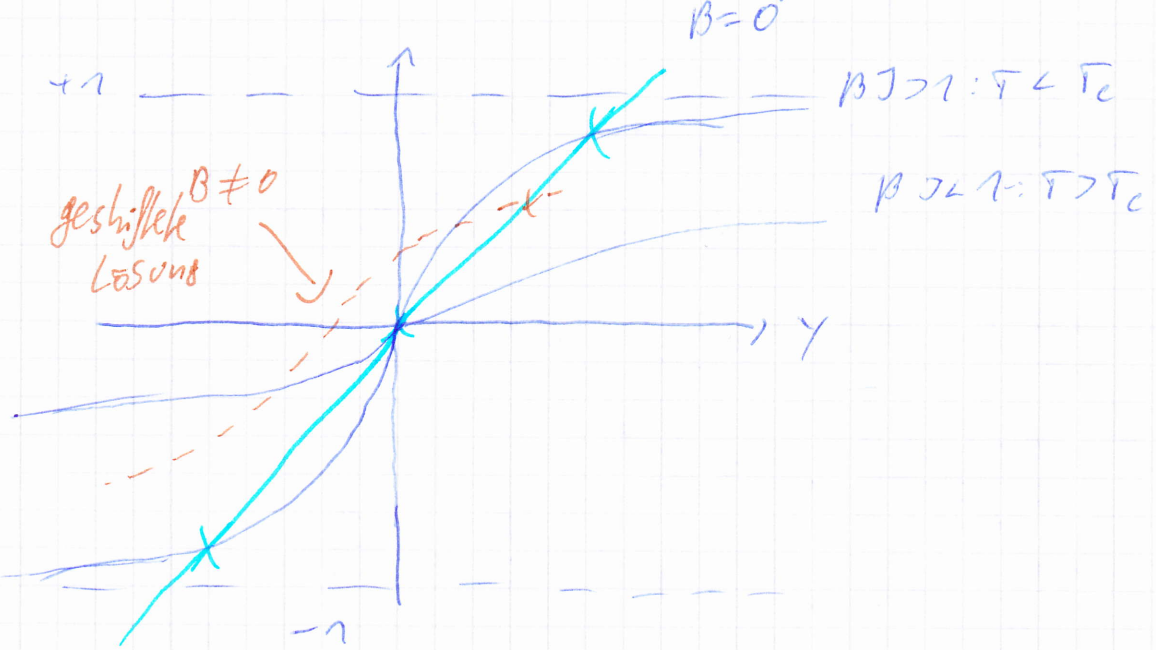
$$\int dx\, e^{-\lambda f(x)} \approx e^{-\lambda f(x_0)} \sqrt{\frac{2\pi}{\lambda f''(x_0)}}$$

$$Z = \sqrt{\frac{\beta J}{f''(y_0)}}\, e^{-N f(y_0, \beta, B)}$$

$$\frac{\ln Z}{N} \hat{=} \frac{1}{N}\sqrt{\dots} - f(y_0,\beta,B) \xrightarrow{N\to\infty} -f(y_0,\beta,B) = -\frac{\beta F}{N}$$

$$y_0 = ?\ ,\quad f' = \beta J y - \frac{2\sinh(..)}{2\cosh(..)}\beta J = \beta J\left[y - \tanh(\beta(Jy+B))\right] \stackrel{!}{=} 0$$

$$y_0 = \tanh(\beta(Jy_0 + B))$$



$$\boxed{T_c := \frac{J}{k}}$$

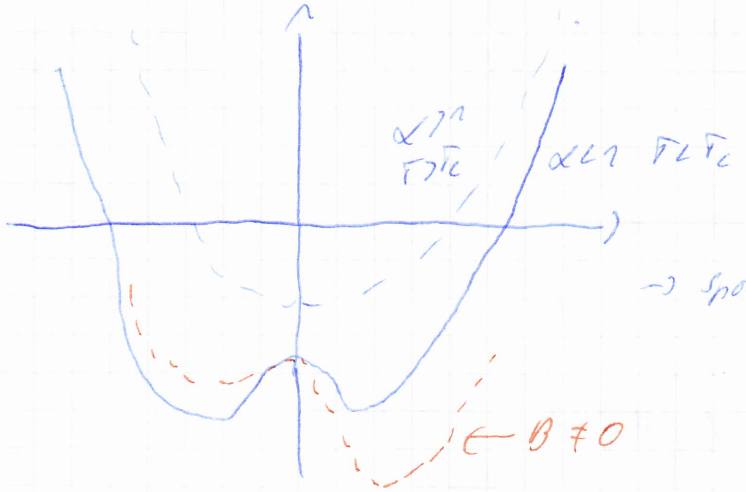
c) $$M = \frac{1}{N}\left\langle \sum_i S_i \right\rangle = \frac{1}{N\beta}\frac{\partial \ln Z}{\partial B}\bigg|_{\beta,N}$$

$$= \frac{1}{N}\frac{1}{Z}\sum_{\{S_i\}}\left(\sum_i S_i\right) e^{-\beta H}$$

$$\rightarrow -\frac{1}{\beta}\frac{\partial f(y_0(B),B)}{\partial B} = -\frac{1}{\beta}\left[\beta J y_0 \frac{\partial y_0}{\partial B} - \underbrace{\tanh\left(\beta(Jy_0+B)\right)}_{y_0}\left\{\beta+\beta J\frac{\partial y_0}{\partial B}\right\}\right]$$

$$= y_0$$

$$f(y) = \frac{y^2}{2\alpha} - \ln 2\cosh\frac{y}{\alpha} \simeq Ay^2 + A'y^4 + \dots$$



$B = 0$

$$\alpha = \frac{1}{\beta J} = \frac{T}{T_c}$$

$\rightarrow$ spontane Symmetriebrechung

d) $$\chi = \frac{\partial M}{\partial B} = \frac{\partial y_0}{\partial B} = \frac{\partial}{\partial B}\tanh\beta(Jy_0+B)$$

$$= \left(1-\tanh^2(\dots)\right)\left(\beta+\beta J\frac{\partial y_0}{\partial B}\right)$$

$$\frac{\partial y_0}{\partial B}\left(1-\beta J(1-y_0^2)\right) = \beta(1-y_0^2)$$

$$\chi = \frac{\beta(1-y_0^2)}{1-\beta J(1-y_0^2)} \sim |\varepsilon|^{\gamma}$$

$T \simeq T_c \qquad \varepsilon = \frac{T-T_c}{T}, \quad \beta J = \frac{T_c}{T} = \frac{1}{1+\varepsilon}$

$\beta = \frac{1}{kT}$

$$\chi = \frac{1}{kT_c}\frac{1}{1+\varepsilon}\frac{1-y_0^2}{1-\frac{1-y_0^2}{1+\varepsilon}} = \frac{1}{kT_c}\frac{1-y_0^2}{\varepsilon+y_0^2}$$

$B=0$

a) $T > T_c$ $(\varepsilon > 0)$: $y_0 = 0 \Rightarrow \chi = \frac{1}{kT_c}\frac{1}{\varepsilon}$, $\gamma = 1$

b) $T < T_c$ $(\varepsilon < 0)$: $y_0 = \tanh\frac{y_0}{1+\varepsilon}$

$T \simeq T_c \Rightarrow y_0$ klein , $\tanh x \simeq x - \frac{x^3}{3}$

$$y_0 \approx \frac{y_0}{1+\varepsilon} - \frac{1}{3}\frac{y_0^3}{(1+\varepsilon)^3} \quad \| \; \frac{1}{y_0}$$

$$\Rightarrow y_0^2 \approx -3\varepsilon + O(\varepsilon^2) = 3|\varepsilon|$$

$$\Rightarrow \chi = \frac{1}{2kT_c}\frac{1}{|\varepsilon|} \Rightarrow \gamma = 1$$