Theo 5 Blatt 1

Nr 1

a) $w_N(n) = \binom{N}{n}\left(\frac{v}{V}\right)^n\left(1-\frac{v}{V}\right)^{N-n}$

$\sum_{n=0}^{N} w_N(n) = \left(\frac{v}{V}+1-\frac{v}{V}\right)^N = 1$ (Binomiallehrsatz)

BLS: $\sum_{n=0}^{N}\binom{N}{n}p^n q^{N-n} = (p+q)^N$

Induktion: $N=0$: $(p+q)^0 = 1 = \binom{0}{0}p^0q^0 = 1$

$N\to N+1$: $(p+q)^{N+1} = p(p+q)^N + q(p+q)^N$

$= p\sum_{n=0}^{N}\binom{N}{n}p^n q^{N-n} + q\sum_{n=0}^{N}\binom{N}{n}p^n q^{N-n}$

$= \sum_{n=1}^{N+1}\binom{N}{n-1}p^n q^{N-n+1} + \sum_{n=0}^{N}\binom{N}{n}p^n q^{N-n+1}$

$= \sum_{n=1}^{N}\binom{N}{n-1}p^n q^{N-n+1} + \sum_{n=1}^{N}\binom{N}{n}p^n q^{N-n+1} + p^{N+1} + q^{N+1}$

$= \sum_{n=1}^{N}\left(\binom{N}{n-1}+\binom{N}{n}\right)p^n q^{N-n+1} + p^{N+1} + q^{N+1}$

$= \sum_{n=1}^{N}\binom{N+1}{n}p^n q^{N-n+1} + \underset{n=0}{p^{N+1}} + \underset{n=N+1}{q^{N+1}}$

$= \sum_{n=0}^{N+1}\binom{N+1}{n}p^n q^{N-n+1}$

b) $\langle n\rangle = \sum_{n=0}^{N} n\, w_N(n)$

$p\frac{\partial}{\partial p}\sum_{n=0}^{N}\binom{N}{n}p^n q^{N-n} = \sum_{n=0}^{N} n\binom{N}{n}p^n q^{N-n} = \sum_{n=0}^{N} n\, w_N(n)$

$= p\frac{\partial}{\partial p}(p+q)^N = pN(p+q)^{N-1} \overset{q+p=1}{=} \underset{\frac{v}{V}}{pN} = \langle n\rangle = \bar{n}$

c) $\frac{\langle(n-\bar{n})^2\rangle}{\bar{n}} = ?$

$\langle(n-\bar{n})^2\rangle = \langle(n-Np)^2\rangle = \sum_{n=0}^{N}(n-Np)^2\binom{N}{n}p^n q^{N-n}$

$\cancel{= Np = Np(p+q) = Np(1-p-q) = Np(}$

$= N^2p^2(p+q)^N - 2Np\,\underbrace{Np + N p^2 (p+q)^{N-2}(N-1)}_{+pN(q+p)^{N-1}}$

$= N p^2\left(N(p+q)^N + (N-1)(p+q)^{N-2} - 2N\right) = N^2p^2\left((p+q)^N + (p+q)^{N-2} - 2N\right) - Np^2(p+q)^{N-2}$

$\frac{\langle(n-\bar{n})^2\rangle}{\bar{n}} = (q+p)^{N-1} + p\left(N(p+q)^N + (p+q)^{N-2} - 2N\right) - p(p+q)^{N-2}$

$= 1 + N\frac{v}{V}(1+1-2) - \frac{v}{V}\cdot 1^{N-2} = 1 - \frac{v}{V}$

$v = V \Rightarrow \frac{\langle(n-\bar{n})^2\rangle}{\bar{n}} = 0$, $v = \frac{V}{2} \Rightarrow \ldots = \frac{1}{2}$, $v = 10^{-10}V \Rightarrow 1-10^{-10}$

$p\frac{\partial}{\partial p}p\frac{\partial}{\partial p}\sum_{n=0}^{N}\binom{N}{n}p^n q^{N-n} = p\frac{\partial}{\partial p}pN(q+p)^{N-1} = pN(q+p)^{N-1} + p^2(N-1)(q+p)^{N-2}$

Nr 2

a/ $p = \frac{N_{pos}}{N_{Ges}}$, $N_{Ges} = N^n$, $N_{pos} = \cancel{\binom{N}{n}} n!$ $\Rightarrow p = \frac{\cancel{\binom{N}{n}} n!}{N^n}$

$\cancel{\Rightarrow p = \frac{\frac{N!}{n!(N-n)!} \cdot n!}{N^n} = \frac{N!}{N^n (N-n)!}}$

b/ $N_{Ges} = \binom{N+n-1}{n}$, $N_{pos} = \cancel{\binom{N}{n}}\ 1$ $\Rightarrow p = \frac{\cancel{\binom{N}{n}}\ 1}{\binom{N+n-1}{n}} = \frac{n!(N+1)!}{(N-\cancel{n}+1)!}$

c/ $N_{Ges} = \binom{N}{n}$, $N_{pos} = 1$ $\Rightarrow$ $p = \frac{1}{\binom{N}{n}} = \frac{n!(N-n)!}{N!}$

Nr. 3

a) $\Gamma(x) = \int_0^\infty dt\, t^{x-1} e^{-t} = \underbrace{\left[-e^{-t} t^{x-1}\right]_0^\infty}_{=0} + \int_0^\infty dt\, (x-1) e^{-t} t^{x-2}$

$\Gamma(x=1) = \int_0^\infty dt\, t^0 e^{-t} = \int_0^\infty dt\, e^{-t} = \left[-e^{-t}\right]_0^\infty = 1 = 0!$

$x \to x+1$ $\Gamma(x+1) = \int_0^\infty dt\, t^{x-1+1} e^{-t} = \underbrace{\left[-e^{-t} t^x\right]_0^\infty}_{=0} + \int_0^\infty dt\, x\, e^{-t} t^{x-1}$

$= \Gamma(x)\big|_{x \to x} \cdot x$

$\Rightarrow$ $\Gamma(x+1) = x!$ n. vollst. Induktion.

b) $I(\lambda) = \int_{t_1}^{t_2} dt\, e^{-\lambda f(t)} \overset{!}{=} \Gamma(x+1) = \int_0^\infty dt\, t^x e^{-t} = \int_0^\infty dt\, e^{-t + x\ln(t)}$

$\Rightarrow \lambda = 1$, $f(t) = t - x\ln(t)$, $f'(t) = 1 - \frac{x}{t}$ $\Rightarrow$ $t = x$ min.

$\Rightarrow I(\lambda) \approx e^{-x + x\ln(x)} \left(2\pi / \frac{1}{x}\right)^{1/2} = e^{-x} x^x \sqrt{2\pi x} = x!$

$f''(t) = \frac{x}{t^2}$