Blatt 4

Aufgabe 11

a) $\Gamma(N, E_0, \delta E_0) = \frac{\partial \Sigma}{\partial E_0} \delta E_0, \quad \Sigma(N, E_0) = \frac{V^N}{h^{3N}} \cdot \frac{(2\pi m E_0)^{3N/2}}{\Gamma\left(\frac{3N}{2}+1\right)}$

$$\frac{\partial \Sigma}{\partial E_0} = \frac{V^N}{h^{3N}} \cdot \frac{(2\pi m)^{3N/2} \cdot \cancel{\frac{3N}{2}} \cdot E^{3N/2-1}}{\underbrace{\Gamma\left(\frac{3N}{2}+1\right)}_{\cancel{\frac{3N}{2}} \cdot \Gamma\left(\frac{3N}{2}\right)}} \Rightarrow \Gamma(N, E_0, \delta E_0) = \frac{V^N}{h^{3N}} \cdot \frac{(2\pi m)^{3N/2} \cdot E^{3N/2-1}}{\Gamma(3N/2)} \delta E_0$$

b) Die Wahrscheinlichkeit, dass die Energie des ersten Teilchens im Intervall $(E_1, E_1 + dE_1)$ liegt:

$$p_1(E_1) = \frac{\Gamma(N-1, E_0 - E_1) \cdot \Gamma(1, E_1)}{\Gamma(N, E_0)}, \quad \text{wobei } \Gamma(N, E) = \frac{\partial \Pi}{\partial E}.$$

$$\Rightarrow p_1(E_1) = \frac{\cancel{V^{N-1}}}{\cancel{h^{3(N-1)}}} \cdot \frac{(2\pi m)^{3(N-1)/2} \cdot (E_0 - E_1)^{3(N-1)/2-1}}{\Gamma\left(\frac{3(N-1)}{2}\right)} \cdot \frac{\cancel{V}}{\cancel{h^3}} \cdot \frac{(2\pi m)^{3/2} \cdot E_1^{1/2}}{\underbrace{\Gamma\left(\frac{3}{2}\right)}_{\sqrt{\pi}/2}} \cdot \frac{\cancel{h^{3N}}}{\cancel{V^N}} \cdot \frac{\Gamma\left(\frac{3N}{2}\right)}{(2\pi m)^{3N/2} \cdot E_0^{3N/2-1}}$$

$$\Rightarrow \boxed{p_1(E_1) = \frac{2}{\sqrt{\pi}} \cdot \frac{\Gamma\left(\frac{3N}{2}\right)}{\Gamma\left(\frac{3(N-1)}{2}\right)} \cdot \frac{(E_0 - E_1)^{3(N-1)/2-1} \cdot E_1^{1/2}}{E_0^{3N/2-1}}}$$

c) $$\int_0^{E_0} p(E_1)\, dE_1 = \frac{2}{\sqrt{\pi}} \cdot \frac{\Gamma(3N/2)}{\Gamma\left(\frac{3(N-1)}{2}\right)} \cdot \frac{1}{E_0^{3N/2-1}} \underbrace{\int_0^{E_0} (E_0 - E_1)^{3(N-1)/2-1} \cdot E_1^{1/2}\, dE_1}_{I}$$

$$y := \frac{E_1}{E_0} \Rightarrow I = \int_0^1 E_0^{3(N-1)/2-1} \cdot E_0^{1/2} \cdot (1-y)^{3(N-1)/2-1} \cdot y^{1/2} \cdot E_0\, dy =$$

$$= E_0^{3N/2-1} \int_0^1 (1-y)^{3(N-1)/2-1} y^{1/2}\, dy \overset{\text{Hinw.}}{=} \frac{\Gamma\left(\frac{3(N-1)}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3N}{2}\right)}$$

$$\Rightarrow \int_0^{E_0} p(E_1)\, dE_1 = \frac{2}{\sqrt{\pi}} \cdot \frac{\Gamma(3N/2)}{\Gamma\left(\frac{3(N-1)}{2}\right)} \cdot \frac{1}{E_0^{3N/2-1}} \cdot E_0^{3N/2-1} \cdot \frac{\Gamma\left(\frac{3(N-1)}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3N}{2}\right)} = 1.$$

d) $\langle E_1\rangle = \int_0^{E_0} dE_1\, E_1\, p(E_1) = \frac{2}{\sqrt{\pi}} \cdot \frac{\Gamma\left(\frac{3N}{2}\right)}{\Gamma\left(\frac{3(N-1)}{2}\right)} \cdot \frac{1}{E_0^{3N/2-1}} \underbrace{\int_0^{E_0} dE_1\, E_1 \cdot (E_0-E_1)^{3(N-1)/2-1} E_1^{1/2}}_{I}$

$$I = \int_0^{E_0} dE_1\, (E_0-E_1)^{3(N-1)/2-1} E_1^{3/2}$$

$$y = \frac{E_1}{E_0} \Rightarrow I = \int_0^1 dy\, E_0 \cdot E_0^{3(N-1)/2-1} \cdot E_0^{3/2} (1-y)^{3(N-1)/2-1} y^{3/2} =$$

$$= E_0^{3N/2} \int_0^1 dy\, (1-y)^{3(N-1)/2-1} y^{3/2} = E_0^{3N/2} \cdot \frac{\Gamma\left(\frac{3(N-1)}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{3N}{2}+1\right)}$$

$$\Rightarrow \langle E_1\rangle = \frac{\cancel{2}}{\cancel{\sqrt{\pi}}} \cdot \frac{\cancel{\Gamma\left(\frac{3N}{2}\right)}}{\cancel{\Gamma\left(\frac{3(N-1)}{2}\right)}} \cdot \frac{1}{\cancel{E_0^{3N/2-1}}} \cdot \cancel{E_0^{3N/2}} \cdot \frac{\cancel{\Gamma\left(\frac{3(N-1)}{2}\right)} \overbrace{\Gamma\left(\frac{5}{2}\right)}^{\frac{3}{2}\Gamma\left(\frac{3}{2}\right)}}{\underbrace{\Gamma\left(\frac{3N}{2}+1\right)}_{\frac{3N}{2}\cdot\Gamma\left(\frac{3N}{2}\right)}} = \frac{E_0}{N}.$$

e) $\langle E_1^2\rangle = \int_0^{E_0} dE_1\, E_1^2\, p(E_1) = \frac{2}{\sqrt{\pi}} \cdot \frac{\Gamma\left(\frac{3N}{2}\right)}{\Gamma\left(\frac{3(N-1)}{2}\right)} \cdot \frac{1}{E_0^{3N/2-1}} \int_0^{E_0} dE_1\, E_1^2 (E_0-E_1)^{3(N-1)/2-1} E_1^{1/2}$

$$\ldots = \frac{5E_0^2}{(3N+2)N} \quad \text{(identisch mit pkt. c), d))}$$

$$\Delta E_1^2 = \langle E_1^2\rangle - \langle E_1\rangle^2 = \frac{5E_0^2}{(3N+2)N} - \frac{E_0^2}{N^2} \overset{!}{=} \frac{E_0^2}{N^2} \cdot \frac{5N-3N-2}{3N+2} \Rightarrow \Delta E_1 = \frac{E_0}{N}\sqrt{\frac{2(N-1)}{3N+2}}$$

f) $p_1(E_1) = \frac{2}{\sqrt{\pi}} \cdot \frac{\Gamma\left(\frac{3N}{2}\right)}{\Gamma\left(\frac{3(N-1)}{2}\right)} \cdot \frac{(E_0-E_1)^{3(N-1)/2-1} E_1^{1/2}}{E_0^{3N/2-1}}$

Stirling Formel: $\Gamma(n) \to \sqrt{2\pi(n-1)} \cdot \left(\frac{n-1}{e}\right)^{n-1}$, für n groß.

$$\Rightarrow \frac{\Gamma\left(\frac{3N}{2}\right)}{\Gamma\left(\frac{3(N-1)}{2}\right)} = \frac{\sqrt{\frac{3N}{2}-1}}{\sqrt{\frac{3N}{2}-\frac{5}{2}}} \cdot \frac{((3N/2-1)/e)^{\frac{3N}{2}-1}}{((3N/2-5/2)/e)^{\frac{3N}{2}-\frac{5}{2}}} = \left(\frac{3N-2}{3N-5}\right)^{\frac{3N}{2}-\frac{1}{2}} \cdot \left(\frac{3N-5}{e}\right)^{3/2} =$$

$$= \sqrt{\frac{3N-2}{3N-5}} \cdot \frac{\left(1-\frac{2/3}{N}\right)^{\frac{3}{2}N-1}}{\left(1-\frac{5/3}{N}\right)^{\frac{3}{2}N-1}} \cdot \left(\frac{3N-5}{2e}\right)^{3/2} \xrightarrow[N\to\infty]{} \frac{e^{-1}}{e^{-5/2}} \cdot \left(\frac{3N}{2}\right)^{3/2} \cdot e^{-3/2} = \left(\frac{3N}{2}\right)^{3/2}$$

[dazu haben wir benutzt, dass $\lim_{x\to\infty} \left(1+\frac{a}{x}\right)^{bx} = \lim_{x\to\infty} \left[e^{bx\ln(1+\frac{a}{x})}\right]$

und $\lim_{x\to\infty} x \ln\left(1+\frac{a}{x}\right) = a$; es folgt $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx} = e^{a\cdot b}$]

$$\Rightarrow \text{für } N\to\infty,\ p_1(E_1) = \frac{2}{\sqrt{\pi}} \cdot \left(\frac{3N}{2}\right)^{3/2} \cdot \left(\frac{E_0-E_1}{E_0}\right)^{\frac{3N-5}{2}} \cdot \frac{E_1^{1/2}}{E_0^{3/2}} =$$

$$= \frac{2}{\sqrt{\pi}} \cdot \left(\frac{3}{2} \cdot \frac{N}{E_0}\right)^{3/2} \cdot \left(1-\frac{E_1}{E_0}\right)^{\frac{3N-5}{2}} \cdot E_1^{1/2}$$

Aus pct e), $\frac{N}{E_0} = \frac{1}{\Delta E_1} \cdot \sqrt{\frac{2(N-1)}{3N+2}} \Rightarrow \frac{N}{E_0} \xrightarrow{N\to\infty} \frac{1}{\Delta E_1} \cdot \sqrt{\frac{2}{3}}$ Use in the following

$$\Rightarrow p_1(E_1) = \frac{2}{\sqrt{\pi}} \cdot \left(\frac{3}{2} \cdot \frac{1}{\Delta E_1} \cdot \sqrt{\frac{2}{3}}\right)^{3/2} \cdot \left(1-\frac{E_1}{E_0}\right)^{\frac{3N-5}{2}} \cdot E_1^{1/2}$$

$$N E_1 \text{ fest} \Rightarrow \left(1-\frac{E_1}{E_0}\right)^{\frac{3N-5}{2}} = \left(1-\frac{N E_1}{N E_0}\right)^{\frac{3N-5}{2}} \xrightarrow{N\to\infty} e^{-\frac{3}{2}\frac{N}{E_0}\cdot E_1}$$

$$\Rightarrow p_1(E_1) = \frac{2}{\sqrt{\pi}} \cdot \left(\frac{1}{\Delta E_1} \cdot \sqrt{\frac{3}{2}}\right)^{3/2} \cdot \exp\left\{-\frac{3}{2} \cdot \frac{1}{\Delta E_1} \cdot \sqrt{\frac{2}{3}} \cdot E_1\right\} \cdot E_1^{1/2}$$

$$\Rightarrow p_1 = \frac{2}{\sqrt{\pi}} \cdot \left(\sqrt{\frac{3}{2}} \cdot \frac{1}{\Delta E_1}\right)^{3/2} \cdot \exp\left\{-\sqrt{\frac{3}{2}} \cdot \frac{E_1}{\Delta E_1}\right\} \cdot E_1^{1/2}$$

(Sorry for the telegraphic writing ...)

Aufgabe 12

a) Im kartesischen Koordinaten:

$$I_n = \int d^n x\, e^{-\vec{x}^2} = \int_{-\infty}^{\infty} dx_1 \ldots dx_n\, e^{-x_1^2 - \ldots - x_n^2} = \int_{-\infty}^{\infty} dx_1\, e^{-x_1^2} \ldots \int_{-\infty}^{\infty} dx_n\, e^{-x_n^2} = (\sqrt{\pi})^n \quad (1)$$

Die Kugelkoordinaten im n Dimensionen folgen aus:

$x_1 = \cos\phi$	$x_1 = \cos\theta$	$x_1 = \cos\theta_1$
$x_2 = \sin\theta$	$x_2 = \sin\theta \cos\phi$	$x_2 = \sin\theta_1 \cos\theta_2$
	$x_3 = \sin\theta \sin\phi$	$x_3 = \sin\theta_1 \sin\theta_2 \cos\phi$
		$x_4 = \sin\theta_1 \sin\theta_2 \sin\phi, \ldots,$

Das Volumenelement: $d^n x = dr\, r^{n-1}\, d\Omega_n$,

$d\Omega_n = (\sin^{n-1}\theta_1\, d\theta_1)(\sin^{n-2}\theta_2\, d\theta_2) \ldots (\sin\theta_{n-1}\, d\theta_{n-1})\, d\phi$,

$\theta_i \in [0, \pi]$; $\phi \in [0, 2\pi)$

$$\Rightarrow I_n = \int d^n x\, e^{-\vec{x}^2} = \int_{S^{n-1}} d\Omega_n \int_0^{\infty} dr\, r^{n-1} e^{-r^2} = S_n \int_0^{\infty} dr\, r\, r^{n-2} e^{-r^2} =$$

$$= S_n \cdot \int_0^{\infty} \frac{dr^2}{2} (r^2)^{\frac{n}{2}-1} e^{-r^2} = \frac{1}{2} S_n \cdot \Gamma\left(\frac{n}{2}\right) \quad (2).$$

Aus (1), (2): $\pi^{n/2} = \frac{1}{2} S_n \cdot \Gamma\left(\frac{n}{2}\right) \Rightarrow \boxed{S_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}}$

Das Volumen: $B_n(R) = \int_{|\vec{x}| \le R} d^n x = S_n \int_0^R dr\, r^{n-1} = S_n \frac{R^n}{n} = R^n \cdot \frac{2}{n} \cdot \frac{\pi^{n/2}}{\Gamma(n/2)} \Rightarrow$

$$\Rightarrow \boxed{B_n(R) = R^n \cdot \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}}$$

Beispiele:

n	1	2	3
$B_n(R)$	$2R$	πR^2	$\frac{4}{3}\pi R^3$
S_n	2	2π	4π

b) Der Phasenraum ist ein Ellipsoid mit Halbachsen $\sqrt{2mE}$ und $\sqrt{\frac{2E}{m\omega^2}}$.

Skaliere die Achsen: $x_i = \sqrt{\frac{2E}{m\omega^2}}\,\tilde{x}_i$, $p_i = \sqrt{2mE}\,\tilde{p}_i$.

$$\Rightarrow W(E) = \int_{\Omega} d^N p\, d^N x = \left(\frac{2E}{m\omega^2}\right)^{N/2} (2mE)^{\frac{N}{2}} \int_{\tilde{\Omega}} d^N \tilde{p}\, d^N \tilde{x} = \left(\frac{2E}{\omega}\right)^N \cdot B_{2N}(1) =$$

$$= \left(\frac{2E}{\omega}\right)^N \cdot \frac{\pi^N}{\Gamma(N+1)} = \left(\frac{2\pi E}{\omega}\right)^N \cdot \frac{1}{N!}.$$

c) In der Quantenmechanik haben wir keine Willkürlichkeit, die Zustände zu zählen. Die Anzahl von Zuständen mit Energie $\hbar\omega(\frac{N}{2}+M)$ wurde in A4 berechnet:

$$\Gamma(N,M) = \frac{(N+M-1)!}{M!\,(N-1)!}$$

Die Anzahl von Zustände mit Energie <u>kleiner</u> als E:

$$\Sigma(E) = \sum_{M=0}^{L} \frac{(N+M-1)!}{M!(N-1)!}, \quad \text{wobei } E \geq \hbar\omega\left(\frac{N}{2}+M\right) \Rightarrow M_{max} \equiv L = \frac{E}{\hbar\omega} - \frac{N}{2}$$

Additionstheorem für die Binomialkoeffizienten:

$$\sum_{k=0}^{m} \binom{n+k}{k} \overset{(*)}{=} \sum_{k=0}^{m} \binom{n+k}{n} = \binom{n+m+1}{n+1} \overset{(**)}{=} \binom{n+m+1}{m}$$

($*$, $**$ folgen aus der Symmetrie der Binomialkoeff.!)

$$\Rightarrow \Sigma(E) = \binom{N+L}{L} \quad \text{(quantenmechanisch!)}$$

Für $E \gg N\hbar\omega$ (also $L \gg N$; N fixiert):

$$\Sigma(E) = \frac{(N+L)!}{N!\,L!} \xrightarrow{L\to\infty} \frac{1}{N!} L^N + O(L^{N-1}) = \frac{1}{N!}\left(\frac{E}{\hbar\omega} - \frac{N}{2}\right)^N + \ldots \simeq$$

$$\simeq \frac{1}{N!}\left(\frac{E}{\hbar\omega}\right)^N = \frac{1}{(2\pi\hbar)^N}\,\frac{1}{N!}\cdot\left(\frac{2\pi E}{\omega}\right)^N \Rightarrow \Sigma(E) \simeq \frac{W(E)}{(2\pi\hbar)^N}.$$