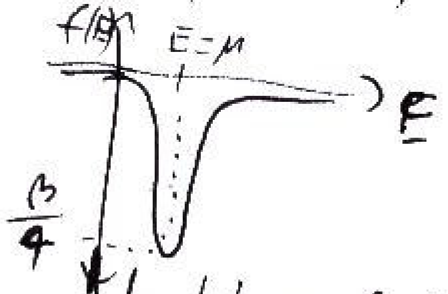
Theo 5 Blatt 9 Julian Beymann

Nr 25

a) $1 = \int_{-\infty}^{\infty} dE\, g(E) f(E) = \underbrace{\rho(E) f(E)\Big|_{-\infty}^{\infty}}_{\underbrace{0\cdot f(-\infty) + \rho(\infty)\cdot \frac{1}{e^{\infty}}}_{=0}} - \int_{-\infty}^{\infty} dE\, \rho(E) \frac{\partial f(E)}{\partial E} = -\int_{-\infty}^{\infty} dE\, \rho(E)\, \partial_E f(E)$

b) $\partial_E f(E) = \frac{-1}{(e^{\beta(E-\mu)}+1)^2} \cdot e^{\beta(E-\mu)} \cdot \beta = -\frac{\beta}{e^{\beta(E-\mu)} + 2 + e^{-\beta(E-\mu)}}$



dom. Beiträge?

Entwicklung: $\rho(E) = \rho(\mu) + \sum_{n=1}^{\infty} \frac{(E-\mu)^n}{n!} \left(\partial_E^n \rho(E)\right)\Big|_{E=\mu}$

$\Rightarrow 1 = -\int_{-\infty}^{\infty} dE \left(\rho(\mu) + \sum_{n=1}^{\infty} \frac{(E-\mu)^n}{n!} \left(\partial_E^n \rho(E)\Big|_{E=\mu}\right)\right) \partial_E f(E)$

$= \underbrace{-\int_{-\infty}^{\infty} dE\, \rho(\mu)\, \partial_E f(E)}_{\rho(\mu) = \int_{-\infty}^{\mu} dE\, g(E)} + \sum_{n=1}^{\infty} \frac{1}{n!} \partial_E^n g(E)\Big|_{E=\mu} \int_{-\infty}^{\infty} dE\, (E-\mu)^n \underbrace{\partial_E f(E)}_{\text{gerade um } \mu}$

$\Rightarrow (E-\mu)^n \to (E-\mu)^{2n}$

$= \int_{-\infty}^{\mu} dE\, g(E) + \sum_{n=1}^{\infty} \frac{1}{(2n)!} \partial_E^{2n-1} g(E)\Big|_{E=\mu} \underbrace{\int_{-\infty}^{\infty} dE\, (E-\mu)^{2n}\, \partial_E f(E)}_{A}$

$A = \int_{-\infty}^{\infty} dx\, x^{2n} \frac{e^x}{(e^x+1)^2} \cdot \frac{1}{\beta^{2n}} = 2 \int_0^{\infty} dz\, \frac{z^{2n}}{\beta^{2n}} \frac{e^z}{(1+e^z)^2}$

$\Rightarrow 1 = \int_{-\infty}^{\mu} dE\, g(E) + \sum_{n=1}^{\infty} \frac{1}{\beta^{2n}} \partial_E^{2n-1} E\Big|_{E=\mu} \cdot 2 \int_0^{\infty} dz\, \frac{z^{2n}}{(2n)!} \frac{e^z}{(1+e^z)^2}$

c) $\int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx = \int_0^{\infty} x^{s-1} \sum_{n=1}^{\infty} e^{-nx} dx = \sum_{n=1}^{\infty} n^{-s} \int_0^{\infty} t^{s-1} e^{-t} dt \Rightarrow \sum_{k=1}^{\infty} k^{-n} \underbrace{\int_0^{\infty} t^{n-1} e^{-t} dt}_{\Gamma(n)} = \int_0^{\infty} \frac{x^{n-1}}{e^x - 1} dx$

$\Rightarrow a_n = \int_0^{\infty} dz\, \frac{z^{2n}}{\Gamma(2n+1)} \frac{e^z}{(1+e^z)^2} = \cancel{\int_0^{\infty} dz\, \frac{z^{2n}}{\Gamma(2n+1)} \frac{1}{e^z - 1}}$

$\cancel{= \int_0^{\infty} dz\, \frac{e^z}{(1+e^z)^2 (e^z - 1)}}$ $\Rightarrow \zeta(2n)(1-2^{1-2n}) = \int_0^{\infty} dz\, \frac{z^{2n-1}}{e^{2n}-1} \cdot \frac{1}{(2n-1)!} \cdot (1-2^{1-2n})$

etc.?

d) $1 = \int_{-\infty}^{\mu} dE\, g(E) + \frac{2(1-2^{-1}) \cdot \frac{\pi^2}{6}}{\beta^2} \partial_E^{2-1} g(E) + \frac{2(1-2^{-3}) \frac{\pi^4}{90}}{\beta^4} \partial_E^{4-1} g(E)$

$= \quad " \quad + \frac{\pi^2}{6\beta^2} \partial_E g(E) + \frac{7\pi^4}{360\beta^4} \partial_E^3 g(E)$

VL $\Rightarrow$ stimmt ...

Nr 26

$$E_F = \left(\frac{6\pi^2}{2s+1}\right)^{2/3} \frac{\hbar^2}{2m} n^{2/3} \quad , \; g(E) : 2s+1 \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{E}$$

$$g(E) = \left(\left(\frac{6\pi^2}{2s+1}\right)^{2/3}\frac{\hbar^2}{2m}n^{2/3}\right)^{-3/2} \cdot \frac{3}{2}\sqrt{E}\,Vn = \frac{3}{2}\sqrt{E}\,E_F^{-\frac{3}{2}}Vn$$

$$U = \int_{-\infty}^{\infty} dE\, g(E) E f(E) = \int_{-\infty}^{\infty} dE\, \frac{3}{2} E\sqrt{E} E_F V n f(E) = \frac{3}{2}E_F^{-\frac{3}{2}} Vn \int_{-\infty}^{\infty} E^{3/2} f(E)\,dE$$

$$= \frac{3}{2}E_F^{-\frac{3}{2}}Vn\left(\int_0^{\mu} dE\, E^{3/2} + \frac{\pi^2}{6}(kT)^2 (E^{3/2})'\big|_\mu + \frac{7\pi^4}{360}(kT)^4 (E^{3/2})'''\big|_\mu\right)$$

$$= \frac{3}{2}E_F^{-\frac{3}{2}}Vn\left(\frac{2}{5}\mu^{5/2} + \frac{\pi^2}{6}(kT)^2 \cdot \frac{3}{2}\sqrt{\mu} + \underbrace{\frac{7}{360}\pi^4 (kT)^4 \frac{3}{2}\cdot\frac{1}{2}\cdot\left(-\frac{1}{2}\right)\mu^{-3/2} + \dots}_{\text{egal}}\right)$$

$$\cancel{= \frac{3}{5}E_F N\left(1 + \frac{5}{72}\pi^2 (kT)^2 \sqrt{\mu}\right) + \dots}$$

$$\cancel{\Omega = -\frac{1}{\beta}\int_{-\infty}^{\infty} dE} \qquad = \frac{3}{5}E_F N\left(\left(\frac{\mu}{E_F}\right)^{5/2} + \frac{5\pi^2}{8}\left(\frac{k_B T}{E_f}\right)^2\left(\frac{\mu}{E_f}\right)^{1/2}\right)$$

$$\left(\frac{\mu}{E_f}\right)^n = 1 - n\frac{\pi^2}{12}\left(\frac{kT}{E_f}\right)^2 \quad \Rightarrow \quad U = \frac{3}{5}E_f N\left(1 - \frac{5\pi^2}{24}\left(\frac{kT}{E_f}\right)^2 + \frac{5\pi^2}{8}\left(\frac{kT}{E_f}\right)^2\left(1 - \frac{\pi^2}{24}\left(\frac{kT}{E_f}\right)^2\right)\right)$$

$$\Rightarrow U = \frac{3}{5}E_f N\left(1 - \frac{5}{12}\pi^2\left(\frac{kT}{E_f}\right)^2\right) + \dots$$

$$\partial_E \ln(1+\exp(-\beta(E-\mu))) = \exp(-\beta(E-\mu))(-\beta) / (1+\exp(-\beta(E-\mu)))$$
$$= \frac{-\beta}{\exp\beta(E-\mu)+1}$$

$$\Rightarrow \Omega = -\frac{1}{\beta}\int_{-\infty}^{\infty} dE\, g(E)\ln\left(1+e^{-\beta(E-\mu)}\right) = -\int_{-\infty}^{\infty} dE\, p(E) f(E)$$

$$= -\frac{3}{2}E_F^{-\frac{3}{2}}N\left(\int_0^\mu \frac{2}{3}E^{3/2}dE + \frac{\pi^2}{6}(kT)^2 \frac{2}{3}(E^{3/2})'\big|_\mu\right) + \dots$$

$$= -\frac{5}{2}E_F N\left(\left(\frac{\mu}{E_F}\right)^{5/2} + \frac{\pi^2}{6}(kT)^2\left(\frac{\mu}{E_F}\right)^{1/2}\right)$$

$$= -\frac{5}{2}E_F N\left(1 - \frac{5}{12}\pi^2\left(\frac{kT}{E_f}\right)^2\right) + \dots \quad (\text{siehe } U)$$

Nr 27

QM-Randbed: $N = \frac{2V}{(2\pi\hbar)^3}\int_0^{p_F} d^3p = \frac{2V}{(2\pi\hbar)^3}\frac{4\pi}{3}(p_F)^3 \Rightarrow p_F = (3\pi^2)^{1/3}\,\hbar\left(\frac{N}{V}\right)^{1/3}$

$\Rightarrow \varepsilon_F = c p_F = (3\pi^2)^{1/3}\hbar c\left(\frac{N}{V}\right)^{1/3}$

$T=0 \Rightarrow$ Alle Niveaus bis Fermienergie

$\Rightarrow E_0(V,N) = N\bar{\varepsilon} = N\frac{\int_0^{p_F} d^3p\, cp}{\int_0^{p_F} d^3p} = \frac{3}{4}N\varepsilon_F$

$U = \frac{3}{5}E_f N$
$= \frac{9}{20}N^2\varepsilon_f$

$\Rightarrow p = -\frac{\partial E_r}{\partial V} = -\sum_{s,p}\partial_V \varepsilon_p \bar{n}_p = \frac{1}{3V}\sum_{s,p}\varepsilon_p \bar{n}_p = \frac{1}{3}\frac{E}{V} \xrightarrow{T=0} \frac{1}{3}\frac{E_0}{V}$