Elektrodynamik-Tutorium

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Inhaltsverzeichnis

1 Elektrodynamik Tutorium vom 29.10.2010

1.1 Partielle Integration

a) $-\frac{d}{dx}e^{-x^2+1} = 2xe^{-x^2+1}$
$\int\limits_0^\infty 2xe^{-x^2+1}dx = [e^{-x^2+1}]_0^\infty = -(0-e) = e$

b) $\int\limits_a^b xe^{-5x}dx = [-\frac{1}{5}e^{-5x}]_a^b - \int\limits_a^b -\frac{1}{5}e^{-5x}dx$

c) $\int\limits_0^\infty x^n e^{-ax}dx = \frac{n!}{a^{n+1}}$
$= \underbrace{[-\frac{1}{a}e^{-ax}x^n]_0^\infty}_{=0} - \int\limits_0^\infty nx^{n-1}(-\frac{1}{a})e^{-ax}dx = \int\limits_0^\infty \frac{n}{a}e^{-ax}x^{n-1}dx$
$= \int\limits_0^\infty \frac{n!}{a^n}e^{-ax}dx + [(-\frac{1}{a})^n e^{-ax}x]_0^\infty = [-\frac{n!}{a^{n-1}}e^{-ax}]_0^\infty = 0 + \frac{n!}{a^{n+1}}$

1.2 δ-Funktion

- $\int\limits_\alpha^\beta f(x)\delta(x-a)dx = f(a)\ \forall \alpha < a < \beta$
- $f(x)\delta'(x-a) = -f'(x)\delta(x-a)$
- $\delta(f(x)) = \sum\limits_i \frac{1}{|f^{-1}(x_i)|}\delta(x-x_i),\ x_i$ NS von $f(x)$
  Bsp.: $x^2-3x+2=0 \Rightarrow x = \{1,2\}$
  $f'(x) = 2x-3,\quad f'(x_1) = 2-3 = -1,\quad f'(x_2) = 4-3 = 1$
  $\int\limits_0^\infty x^2\delta(x^2-3x+2)dx = \int\limits_0^\infty (\frac{1}{|-1|}x^2\delta(x-1) + \frac{1}{|1|}x^2\delta(x-2))dx = 1+4 = 5$

Beispiele: $\int\limits_{-\infty}^{+\infty} dx(x^2+7x)\delta(x-x_0) = x_0^2 + 7x_0$

Behauptung: $\delta(\vec{r}-\vec{r}') = \frac{1}{4\pi}\Delta_r\frac{1}{|\vec{r}-\vec{r}'|}$, Beweis:Nolting

Beweisskizze: Es muss gelten $\delta(x-a) = 0,\ \forall a \neq x$ und $\int\limits_\alpha^\beta dx\delta(x-a) = 1$

$\Delta_r = \vec{\nabla}_r \cdot \vec{\nabla}_r,\quad \vec{\nabla}\vec{A} = 0 \Rightarrow$ Quellenfrei, $\vec{\nabla}\times\vec{A} = 0 \Rightarrow$ Wirbelfrei

1.3 Beispiele

a) $\int\limits_0^1 \sqrt{1-x^2}dx,\quad x = \sin(t),\quad dx = \cos(t)dt$
$\Rightarrow \int\limits_{\arcsin(0)}^{\arcsin(1)} \sqrt{1-\sin^2(t)}\cos(t)dt = \int\limits_0^{\pi/2} \sqrt{\cos^2(t)}\cos(t)dt = \int\limits_0^{\pi/2} \cos^2(t)dt$
$\cos^2(t) = \frac{1+\cos(2t)}{2} \Rightarrow ... = [\frac{x}{2} + \frac{1}{4}\sin(2x)]_0^{\pi/2} = \frac{\pi}{4}$

b) $\int\limits_0^2 x\cos(x^2+1)dx,\quad t = x^2+1,\quad 2xdx = dt$
$... = \int\limits_1^5 \frac{1}{2}\cos(t)dx = \frac{1}{2}[\sin(t)]_1^5$

2 Elektrodynamik Tutorium vom 5.11.2010

2.1 Integral-Grenzwerte

- $\int\limits_0^\infty x^2 e^{-2x} dx = [(-\frac{1}{2}x^2 - \frac{1}{2}x - \frac{1}{4})e^{-2x}]_0^\infty = 0 + \frac{1}{4}$
  $\int\limits_0^\infty x^2 e^{-2x} dx = \frac{n!}{a^{n+1}} = \frac{1}{4}$
- $\int\limits_1^\infty x e^{x^2+1} dx, \quad t = x^2 - 1$
  $\Rightarrow ... = \int\limits_0^\infty \frac{x}{2x} e^{-t} dt = \frac{1}{2} \int\limits_0^\infty e^{-t} dt = \frac{1}{2}$
- $\int\limits_{-\infty}^\infty (3x^2 + 5x)\delta(x-3)dx = (3x^2+5x)|_{x=3} = 42$
- $\int\limits_{-5}^5 (x^2+x)\delta(2x^2 - 4x - 6)dx$
  $0 = 2x^2 - 4x - 6 \Rightarrow x_1 = -1, x_2 = 3$
  $f'(x) = 4x - 4), \; f'(x_1) = 8, \; f'(x_2) = -8$
  $\Rightarrow ... = \int\limits_{-5}^5 \frac{1}{8}(x^2+x)(\delta(x-3) + \delta(x+1))dx = \frac{12}{8}$

2.2 Beispiele

$r := \sqrt{a^2 + b^2 + c^2}$
$|\vec{a} - \vec{b}| = \sqrt{a^2 + b^2 - 2ab\cos(\theta)}$
$\varphi(r + r')$ um r: $\varphi(r+r') = \varphi(x_1 + x_1', x_2 + x_2', x_3 + x_3') = F(t = 1)$
$F(t) = \varphi(x_1 + x_1' t, x_2 + x_2' t, x_3 + x_3' t)$
$F(t) = \sum\limits_{n=0}^\infty \frac{1}{n!} F^{(n)}(0) t^n$ um $t = 0$
$F'(0) = \sum\limits_{j=1}^3 \frac{\partial\varphi}{\partial x_j} x_j$
$F''(0) = \sum\limits_{jk} x_j' x_k' \frac{\partial^2}{\partial x_k \partial x_j} \varphi(r) = (\sum\limits_j^{x_j \frac{\partial}{\partial x_j}})^2 \varphi(r)$
$F^{(n)}(0) = (\sum\limits_j^{x_j \frac{\partial}{\partial x_j}})^n \varphi(r)$
$F(t=1) = varphi(r+r') = \sum\limits_{n=0}^\infty \frac{1}{n!} (\sum\limits_j x_j \frac{\partial}{\partial x_j})^n \varphi(r)$

Beispiel:
$\varphi(r) = \frac{\alpha}{|r - r_0|}$ um r=0 entwickeln:
$n = 0: \; \varphi(r)|_{r=0} = \frac{\alpha}{r_0}, r_0 > 0$
$n = 1: \; \sum\limits_{j=1}^3 x_j \frac{\partial}{\partial x_j} \varphi(r)|_{r=0} = \sum\limits_{j=1}^3 x_j \frac{\partial}{\partial x_j} (\frac{\alpha}{|r - r_0|})_{r=0} = \sum\limits_{j=1}^3 x_j x(j_0) \frac{\alpha}{r_j^3}$
$n = 2: \; \sum\limits_{j,k} x_j x_k \frac{\partial^2}{\partial x_k \partial x_j} \varphi(r)$
$\frac{\partial^2}{\partial x_k \partial y_k} = -\frac{\partial}{\partial x_k} \frac{\alpha(x_j - x_{j_0})}{|r - r_0|^2}$

$$\sum_{j,k} x_j x_k \frac{\partial^2}{\partial x_k \partial x_j}\varphi(r) = \sum_{j,k} x_j x_k \frac{\partial}{\partial x_k}\left(\frac{\alpha(x_j - x_0)}{|r-r_0|}\right) = \sum_{j,k} x_j x_k \left(\delta_{jk}\frac{-\alpha}{|r-r_0|^3} + \frac{3\alpha(x_j - x_{j0})(x_k - x_{k0})}{|r-r_0|^5}\right)$$
$$= \sum_{j,k} x_j x_k \left(-\frac{\alpha\delta_{kj}}{r_0^3} + \frac{3\alpha x_{j0} x_{k0}}{r_0^5}\right) = \alpha\left(\frac{3(r-r_0)}{r_0^3} - \frac{r^2}{r_0^3}\right)$$

2.3 Satz von Gauß

$$\oint_{S(V)} \vec{E} d\vec{f} = \int_V \vec{v}\vec{E} dV$$

$\vec{A} = (3xy, y, 0)$ über Halbkugel r=1 + Grundfläche

$\vec{\nabla}\vec{A} = 3y + 1$

$$\int_V 3y + 1 dV = \int_0^{\pi/2}\int_0^{\pi}\int_0^{r} (3r\sin(\vartheta)\sin(\varphi) + 1) r^2 \sin(\vartheta) dr d\varphi d\vartheta = \int_0^{\pi/2}\int_0^{1} 2\pi r^2 \sin(\vartheta) dr d\vartheta$$
$$= 2\pi \int_0^1 r^2 dr = \frac{2\pi}{3}$$

$$\int \vec{A}\vec{e}_r d\vec{r} = \int_0 \vec{e}_r \vec{A} dF$$

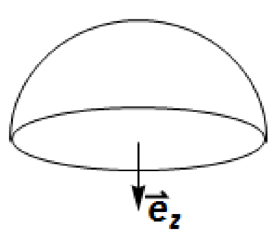
$$\vec{e}_r\vec{A} = \begin{pmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix} \begin{pmatrix} 3\sin^2\vartheta\cos\varphi\sin\varphi \\ \sin\vartheta\sin\varphi \\ 0 \end{pmatrix}$$

3 Elektrodynamik Tutorium vom 12.11.2010

3.1 Beispiele

- $\int_{-\infty}^{1} 3xe^{-(x^2-1)}dx$ mit $t = x^2 - 1$, also $\frac{dt}{dx} = 2x$, $\frac{dt}{2x} = dx$
  $= \int_{\infty}^{0} \frac{3}{2} e^{-t} dt = [-\frac{3}{2}e^{-t}]_{\infty}^{0} = -\frac{3}{2}$
- $\int_0^5 (3x^2 - \frac{1}{x^2})\delta(x-3)dx = 27 - \frac{1}{9}$
- $\vec{\nabla}\vec{r} = 3$
- $\vec{A} = (3xy, x^2, x^2 + y^2)$
  $\vec{\nabla}\vec{A} = 3y$
  $\vec{\nabla} \times \vec{A} = (2y, -2x, -x)$

3.2 Fortsetzung: Halbkugel mit R=1



$\vec{A} = (3xy, y, 0)$, $\int_V \vec{\nabla}\vec{A} dV = \frac{2}{3}\pi$

$$\int_V \vec{\nabla} dV = \oint_{\partial V} \vec{A} d\vec{F}$$

$$\oint_{\partial V} \vec{A} dF = \int_{\cap} \vec{e}_r \vec{A} dF - \int_0 \vec{e}_z \vec{A} dS = \int_0^{\pi/2}\int_0^{2\pi} \vec{e}_r \sin\vartheta \vec{A} R^2 d\vartheta d\varphi$$

$$\vec{e}_r\vec{A} = \begin{pmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix} \begin{pmatrix} 3\sin^2\vartheta\cos\varphi\sin\varphi \\ \sin\vartheta\sin\varphi \\ 0 \end{pmatrix} = 3\sin^3\vartheta\cos^2\varphi\sin\varphi + \sin^2\vartheta\sin^2\varphi$$

$\frac{d\cos\varphi}{d\varphi} = -\sin\varphi \Leftrightarrow d\varphi = -\frac{d\cos\varphi}{\sin\varphi}$

$\Rightarrow \int\limits_0^{\pi/2} \underbrace{\int\limits_0^{2\pi} (-3\sin^4\vartheta\cos^2\varphi d\cos\varphi)}_{-\frac{1}{3}\cos^3\varphi|_0^{2\pi}} d\vartheta + \int\limits_0^{\pi/2}\int\limits_0^{2\pi} \sin^3\vartheta\sin^2\varphi d\varphi d\vartheta = \pi\int\limits_0^{\pi/2}\sin^3\vartheta d\vartheta = \frac{2}{3}\pi$

3.3 Koordinatensysteme

$\vec{e}_1, \vec{e}_2, \vec{e}_3$

$\vec{r} = \sum\limits_{j=1}^{3} x_j\vec{e}_j$

$d\vec{r} = \sum\limits_{i=1}^{3} dx_j\vec{e}_j = \sum\limits_{i=1}^{3}\frac{\partial r}{\partial x_j}dx_j$

y_1, y_2, y_3

$\vec{e}_{y_j} = \frac{\frac{\partial\vec{r}}{\partial y_j}}{|\frac{\partial\vec{r}}{\partial y_j}|}$

$e_{y_j}\cdot e_{y_i} = \delta_{ij}$

Beispiel: $x = r\cos\varphi,\ y = r\sin\varphi$

$e_r = \frac{\frac{\partial\vec{r}}{\partial r}}{|\frac{\partial\vec{r}}{\partial r}|},\quad \frac{\partial\vec{r}}{\partial r} = \begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}$

$e_r = \begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}$

$e_\varphi = \frac{1}{j}\begin{pmatrix}-r\sin\varphi\\ r\cos\varphi\end{pmatrix} = \begin{pmatrix}-\sin\varphi\\ \cos\varphi\end{pmatrix},\ \frac{\partial\vec{r}}{\partial\varphi} = \begin{pmatrix}-r\sin\varphi\\ r\cos\varphi\end{pmatrix}$

$J = \frac{\partial(x,y)}{\partial(r,\varphi)} = \begin{vmatrix}\cos\varphi & -r\sin\varphi\\ \sin\varphi & -r\cos\varphi\end{vmatrix} = r\cos^2\varphi + r\sin^2\varphi = r$

$dV = dx_1dx_2 = rdrd\varphi$

3.4 Beispiel: parabolische Koordinaten

$x = \frac{1}{2}(u^2 - v^2),\quad y = u\cdot v,\quad z' = z$

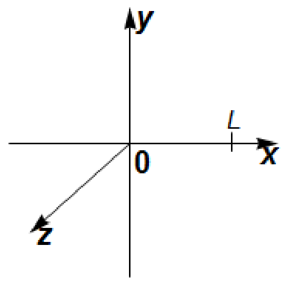
$\vec{e}_u = \frac{1}{\sqrt{u^2+v^2}}matcuv0$

$\vec{e}_v = \frac{1}{\sqrt{u^2+v^2}}matc{-vu0}$

$\vec{e}_z = matc001$

$J = \begin{vmatrix}u & -v & 0\\ v & u & 0\\ 0 & 0 & 1\end{vmatrix} = u^2 + v^2 \Rightarrow dV = (u^2+v^2)dudv$

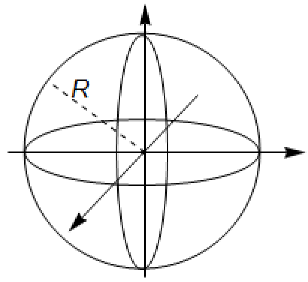
3.5 Ladungsverteilung Draht



$\rho(\vec{r}) = \frac{Q}{L}\delta(x)\delta(z)\theta(L-x)$

$\theta(x - x_0) = \begin{cases}0 & x - x_0 < 0\\ 1 & x - x_0 > 0\end{cases}$

3.6 Ladungsverteilung Kugelschale



Radius R, Gesamt-Ladung Q.

$\rho(\vec{r}) = \frac{Q}{4\pi R^2}$

$\int\limits_V \vec{\nabla}\vec{E}d^3r = \int\limits_V \frac{\rho(\vec{r}')}{\varepsilon_0}d^3r' = \oint\limits_{\partial V} \vec{E}\vec{n}dF$

$\int\limits_V \frac{\rho(\vec{r}')}{\varepsilon_0}d^3r' = \frac{Q}{4\pi R^2\varepsilon_0} = \int\limits_0^{pi}\int\limits_0^{2\pi}\int\limits_0^{R_a} \delta(r'-R)r'^2\sin\vartheta dr'd\varphi d\vartheta = \frac{Q}{R^2\varepsilon_0}R^2 = \frac{Q}{\varepsilon_0}$

$\vec{E}(\vec{r}) = E(r)\vec{e}_r$

$\oint\limits_{\partial V} \vec{E}\vec{n}dF = \oint\limits_{\partial V} E(r)\vec{e}_r\vec{e}_r dF = E(r)\int\limits_0^{\pi}\int\limits_0^{2\pi} R^2\sin\vartheta d\vartheta d\varphi = 4\pi R^2 E(r)$

$$E(r) = \begin{cases} \frac{Q}{4\pi\varepsilon_0 R^2} & \text{für } r > R \\ 0 & \text{für } r \leq R \end{cases}$$

4 Elektrodynamik Tutorium vom 19.11.2010

4.1 Integrale

- $\int\limits_{-\pi}^{\pi} \sin\theta\cos\theta d\theta : \sin\theta = x|\frac{d\sin\theta}{d\theta} = \cos\theta \Rightarrow ... = \int\limits_0^0 xdx = 0$
- $\int\limits_0^{\pi} \sin xe^x dx = \sin xe^x|_0^{\pi} - \int\limits_0^{\pi} \cos xe1xdx = -\cos xe^x|_0^{\pi} + \int\limits_0^{pi} t\sin xe^x dx$

  $2\int\limits_0^{\pi} \sin xe^x dx = -\cos xe^x|_0^{\pi} = e^{\pi}+1$

4.2 Fortsetung E-Feld Kugelschale

$\rho(\vec{r}) = \frac{Q}{2\pi R}\delta(r'-R)$

$\int\limits_V \vec{\nabla}\vec{E}d^3r = \int\limits_V \frac{\rho(\vec{r}')}{\varepsilon_0}d^3r' = \oint\limits_{S(V)} \vec{E}\vec{n}dF = E(r)4\pi r^2$

$$\int\limits_V \frac{\rho(\vec{r}')}{\varepsilon_0}d^3r' = \frac{Q}{\varepsilon_0} \Rightarrow E(r) = \begin{cases} \frac{Q}{4\pi\varepsilon_0 r^2} & \forall r > R \\ 0 & \forall r < R \end{cases}$$

4.3 Poisson-Gleichungen

$\Delta\varphi(r) = -\frac{\rho(\vec{r})}{\varepsilon_0}$

Randbedingungen: $S(V), \varphi$ oder $\frac{\partial\varphi}{\partial n}$

$$\Rightarrow \varphi(r) = \underbrace{\frac{1}{4\pi\varepsilon_0}(\int\limits_V d^3r'\frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}) + \frac{1}{4\pi}}_{Ladung\ in\ Volumen} + \underbrace{\frac{1}{4\pi}\int\limits_{S(V)} df'[\frac{1}{|\vec{r}-\vec{r}'|}\frac{\partial\varphi}{\partial n'} - \varphi(r')\frac{\partial}{\partial n'}\frac{1}{|\vec{r}-\vec{r}'|}]}_{Ladung\ auf\ Oberfläche}$$

Dirichlet: φ auf S(V) (meistens)

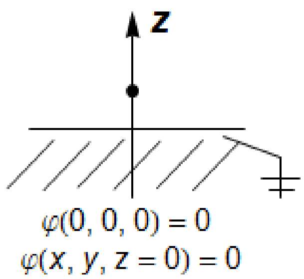
Neumann: $\frac{\partial\varphi}{\partial n}$ auf S(V) (selten, Feld senkrecht zur Oberfläche ändernd... hier nicht)

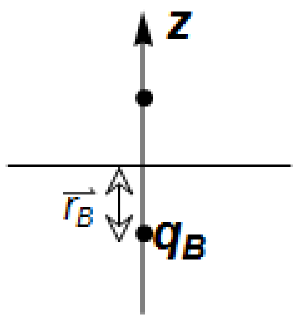
Green'sche Fkt: $G(\vec{r},\vec{r}') = \frac{1}{4\pi\varepsilon_0}\frac{1}{|\vec{r}-\vec{r}'|} + f(\vec{r},\vec{r}')$ q=1

Es muss gelten: $\Delta_r f(\vec{r},\vec{r}') = 0$

$D: \int\limits_{S(V)} df' G_D(\vec{r}',\vec{r})\frac{\partial\varphi}{\partial n'} = 0$

$G_D(\vec{r}'\vec{r}) = \frac{1}{4\pi\varepsilon_0}(\frac{q}{|\vec{r}'-\vec{r}|} + \frac{q_B}{|\vec{r}-\vec{r}_B|})$





$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int\limits_V d^3r' \rho(\vec{r}')(\frac{1}{|\vec{r}'-\vec{r}|} + \frac{1}{|\vec{r}-\vec{r}_B|})$

Ansatz: $4\pi\varepsilon_0\varphi(r) = \frac{q}{|\vec{r}'-\vec{r}|} + \frac{q_B}{|\vec{r}-\vec{r}_B|} \stackrel{!}{=} 0$

$\Rightarrow q_B = -q \Rightarrow z_B = -z \Rightarrow \vec{r}_B = -\vec{r}$

$\Rightarrow 4\pi\varepsilon_0\varphi(r) = \frac{q}{|\vec{r}-\vec{r}'|} - \frac{q}{|\vec{r}+\vec{r}'|}$

$E(r) = -\vec{\nabla} r\varphi(\vec{r}, \vec{r}')$

$\varphi(r) = \frac{q}{4\pi\varepsilon_0}(\frac{1}{|\vec{r}-\vec{r}'|} - \frac{1}{|\vec{r}+\vec{r}'|}) \Rightarrow \vec{r}'(0, 0, z)$

$\sigma = \varepsilon_0 E(r, z = 0)$, $\overline{q} = \int \sigma dF$

4.4 Fourier

$\int\limits_a^b f^*(x)f(x)dx = ||f(x)||^2 = \int\limits_a^b |f(x)|^2 dx < \infty$

$\{f_n(x)\}_{n=1,2,\ldots}$, orthogonal: $\int\limits_a^b f_m^*(x)f_n(x)dx = \delta_{mn}$

$f(x) = \frac{1}{2}A_0 + \sum\limits_{n=1}^{\infty}[A_n \cos\frac{2\pi nx}{a} + B_n \sin\frac{2\pi nx}{a}]$

$A_n = \frac{2}{a}\int\limits_{-a/2}^{a/2} f(x)\cos\frac{2\pi nx}{a}dx, \quad B_n = \frac{2}{a}\int\limits_{-a/2}^{a/2} f(x)\sin\frac{2\pi nx}{a}dx$

5 Elektrodynamik Tutorium vom 26.11.2010

5.1 Kugelfunktion

$\phi(r, \vartheta, \varphi) = \sum\limits_{l=0}^{\infty}\sum\limits_{m=-l}^{l}(a_{lm}r^l + \frac{b_{lm}}{r^{l+1}})Y_{lm}(\vartheta, \varphi)$

Allg Lösung von $\Delta\phi = 0$

5.2 Integrale

- $\int\limits_0^1 \frac{1}{x^3}ln(x^2)dx$

  $\frac{1}{x^2} = a = x^{-2} \Leftrightarrow \frac{da}{dx} = -2x^{-3}, ln(\frac{1}{a}) = -ln(a)$

  $\Rightarrow \int\limits_{\infty}^{1} \frac{1}{2}ln(a)da = \frac{1}{2}[-a + aln[a]]_1^{\infty} =...$

5.3 Hausaufgabenhinweise

gerade Funktion: $f(x) = f(-x) \Rightarrow \cos(x)$

ungerade Funktion: $-f(x) = f(-x) \Rightarrow \sin(x)$

$\int\limits_a^b f^*(x)f(x)dx = ||f(x)||^2 = \int\limits_a^b |f(x)|^2 dx = N$

$\int\limits_a^b \frac{1}{\sqrt{N}}f^*(x)\frac{1}{\sqrt{N}}f(x)dx = \frac{N}{N} = 1$

Fourier-Reihe:

$$f(x) = \frac{1}{2}A_0 + \sum\limits_{n=1}^{\infty}[A_n\cos(\frac{2\pi nx}{a}) + B_n\sin(\frac{2\pi nx}{a})]$$

$$A_n = \frac{2}{a}\int\limits_{-a/2}^{a/2} f(x)\cos(\frac{2\pi nx}{a})dx$$

$$B_n = \frac{2}{a} \int\limits_{-a/2}^{a/2} f(x) \sin(\frac{2\pi n x}{a}) dx$$

Fourier-Funktion:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} dk \tilde{f}(k) e^{ikx}$$

$$\tilde{f}(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} dx f(x) e^{-ikx}$$

Beispiel: $f(x) = e^{-x^2/2}$

$$\tilde{f}(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} dx e^{-x^2/2} e^{-ikx}$$

Hinweis: $\int\limits_{-\infty}^{\infty} dx e^{-x^2/a} = \pi$

$$x^2 + 2ikx + (ik)^2 - (ik)^2 = (\underbrace{x + ik}_{z})^2 + k^2$$

$$\Rightarrow \tilde{f}(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} dx e^{\frac{-x^2 - 2ikx}{2}}$$

$$= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} dz e^{-z^2/2} e^{-k^2/2}$$

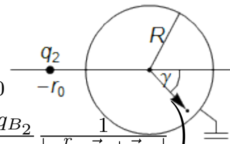
$$= \frac{1}{\sqrt{2\pi}} e^{-k^2/2} \int\limits_{-\infty}^{\infty} e^{-z^2/2} dz = \frac{\sqrt{2\pi}}{\sqrt{2\pi}} e^{-k^2/2} = e^{-k^2/2}$$

5.4 geerdete Kugel

Dirichlet: da Oberfläche geerdet $(\varphi(\vec{r})|_{r=R} = 0)$

$$\varphi(\vec{r}) = \frac{q_1}{4\pi\varepsilon_0} \frac{1}{|\vec{r} - \vec{r}_0|} + \frac{q_2}{4\pi\varepsilon_0} \frac{1}{|\vec{r} + \vec{r}_0|} + f(\vec{r}, \vec{r}_0)$$

$$= \frac{1}{4\pi\varepsilon_0} \left(\frac{q_1}{|\vec{r} - \vec{r}_0|} + \frac{q_2}{|\vec{r} + \vec{r}_0|} + \frac{q_{B_1}}{|\vec{r} - \vec{r}_{B_1}|} + \frac{q_{B_2}}{|\vec{r} - \vec{r}_{B_2}|} \right)$$



$$\vec{r} = r\vec{e}_r, \quad \vec{r}_0 = r0\vec{e}_{r_0}, \quad \vec{r}_{B_1} = r_{B_1}\vec{e}_{r_0}, \quad \vec{r}_{B_2} = r_{B_2}\vec{e}_{r_0} \Rightarrow \vec{r}_B || \vec{r}_0$$

$$\varphi(r) = \frac{1}{4\pi\varepsilon_0} \left(\frac{q_1}{r} \frac{1}{|\vec{e}_r - \frac{r_0}{r}\vec{e}_{r_0}|} + \frac{q_2}{r} \frac{1}{|\vec{e}_r - \frac{r_0}{r}\vec{e}_{r_0}|} + \frac{q_{B_1}}{r_{B1}} \frac{1}{|\frac{r}{r_{B_1}}\vec{e}_r - \vec{e}_{r_0}|} + \frac{q_{B_2}}{r_{B_2}} \frac{1}{|\frac{r}{r_{B_2}}\vec{e}_r + \vec{e}_{r_0}|} \right)$$

$$= \varphi(r) = \frac{1}{4\pi\varepsilon_0} \left(\frac{q_1}{r} \frac{1}{(1 + \frac{r_0^2}{r^2} - 2\frac{r_0}{r}\cos(\gamma))^{1/2}} + \frac{q_2}{r} \frac{1}{(1 + \frac{r_0^2}{r^2} - 2\frac{r_0}{r}\cos(\gamma))^{1/2}} + \frac{q_{B_1}}{r_{B1}} \frac{1}{(1 + \frac{r^2}{r_{B_1}^2} - 2\frac{r}{r_{B_1}}\cos(\gamma))^{1/2}} + \frac{q_{B_2}}{r_{B_2}} \frac{}{(1 + \frac{r^2}{r_{B_2}^2} + 2} \right.$$

$$\Rightarrow -\frac{q_1}{R} = \frac{q_{B_1}}{r_{B_1}} \Rightarrow q_{B1} = -q_1 \frac{r_{B_1}}{R}$$

$$\Rightarrow -\frac{q_2}{R} = \frac{q_{B_2}}{r_{B_2}} \Rightarrow q_{B2} = -q_2 \frac{r_{B_2}}{R}$$

$$1 + \frac{r_0^2}{R^2} - 2\frac{r_0}{R}\cos(\gamma) = 1 + \frac{R^2}{r_{B_1}^2} - 2\frac{R}{r_{B_1}}\cos(\gamma)$$

$$\Rightarrow \frac{r_0}{R} = \frac{R}{r_{B_1}} \Rightarrow r_{B_1} = \frac{R^2}{r_0}$$

$$\Rightarrow r_{B_2} = \frac{R^2}{r_0}$$

$$\Rightarrow q_{B_1} = -q_1 \frac{R}{r_0}, \quad q_{B_2} = -q_2 \frac{R}{r_0} \Rightarrow \varphi(r) = \frac{1}{4\pi\varepsilon_0} \left(\frac{q_1}{|\vec{r} - \vec{r}_0|} + \frac{q_2}{|\vec{r} + \vec{r}_0|} - \frac{q_1}{|\vec{r} - \frac{R^2}{r_0^2}\vec{r}_0|} \frac{R}{r_0} - \frac{q_2}{|\vec{r} + \frac{R^2}{r_0^2}\vec{r}_0|} \frac{R}{r_0} \right)$$

6 Elektrodynamik Tutorium vom 3.12.2010

6.1 Anwendung Nabla in Kugelkoord

$$\Delta\phi = \vec{\nabla}^2\phi = 0,$$

$$\Delta = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r) + \frac{1}{r^2\sin(\vartheta)}\frac{\partial}{\partial\vartheta}(\sin(\vartheta)\frac{\partial}{\partial\vartheta}) + \frac{1}{r^2\sin^2(\vartheta)}\frac{\partial^2}{\partial\vartheta^2}$$

$$\Delta\phi = \vec{\nabla}^2\phi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\phi) + \frac{1}{r^2\sin(\vartheta)}\frac{\partial}{\partial\vartheta}(\sin(\vartheta)\frac{\partial\phi}{\partial\vartheta}) + \frac{1}{r^2\sin^2(\vartheta)}\frac{\partial^2\phi}{\partial v^2} = 0$$

Ansatz: $\phi : (r, \vartheta, \varphi) = u(r)g(\vartheta)\chi(\varphi)$

$$0 = \frac{g\lambda}{r}\frac{\partial^2 f}{\partial r^2} + \frac{fg}{r^3\sin^2(\vartheta)}\frac{\partial^2\chi}{\partial\varphi^2} + \frac{f\chi}{r^3\sin\vartheta}\frac{\partial}{\partial\vartheta}(\sin(\vartheta)\frac{\partial g}{\partial\vartheta}) \quad | \cdot \frac{r^3\sin^2(\vartheta)}{fg\chi}$$

$$0 = \frac{r^2\sin^2(\vartheta)}{f}\frac{\partial^2 f}{\partial r^2} + \underbrace{\frac{1}{\chi}\frac{\partial^2\chi}{\partial\varphi^2}}_{-m^2=const} + \frac{\sin(\vartheta)}{g}\frac{\partial}{\partial\vartheta}(\sin(\vartheta)\frac{\partial g}{\partial\vartheta})$$

$$\frac{1}{\chi}\frac{\partial^2\chi}{\partial\varphi^2}+m^2 = 0 \Leftrightarrow \frac{\partial^2\chi}{\partial\varphi^2}+\chi m^2 = 0 \Rightarrow \chi(\varphi) = e^{\pm imp} \quad \frac{m^2}{\sin^2\vartheta} = \underbrace{\frac{r^2}{f}\frac{\partial^2 f}{\partial r^2}}_{=:l(l+1)} + \frac{1}{g\sin(\vartheta)}\frac{\partial}{\partial\vartheta}(\sin(\vartheta)\frac{\partial g}{\partial\vartheta})$$

$$r^2\frac{1}{f}\frac{\partial^2 f}{\partial r^2} - l(l+1) = 0$$

$$\frac{\partial^2 f}{\partial r^2} - \frac{f}{r^2 l}(l+1) = 0$$

$$f(r) = r^\gamma \Leftrightarrow \gamma(\gamma-1) = l(l+1) \Rightarrow \gamma = -l \ \vee \ \gamma = l+1$$

$$f(r) = A_l r^{l+1} + B_l r^{-l}$$

$$\Rightarrow 0 = l(l+1) - \frac{m^2}{\sin^2(\vartheta)} + \frac{1}{g\sin(\vartheta)}\frac{\partial}{\partial\vartheta}(\sin(\vartheta)\frac{\partial}{\partial\vartheta})$$

$$= (l(l+1) - \frac{m^2}{\sin^2(\vartheta)})g + \frac{1}{\sin(\vartheta)\frac{\partial}{\partial\vartheta}(\sin(\vartheta)\frac{\partial g}{\partial\vartheta}}$$

$$\Rightarrow x = \cos(\vartheta), \quad d\vartheta = -\frac{dx}{\sin(\vartheta)}$$

$$\Rightarrow 0 = \frac{d}{dx}((1-x^2)\frac{dg}{dx}) + [l(l+1) - \frac{m^2}{1-x^2}]g$$

$$g(x) = \sum_{n=0}^{\infty} c_n x^n$$

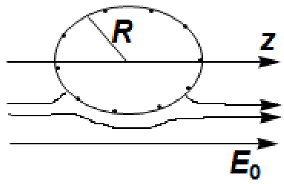
$$\phi(r,\vartheta,\varphi,x=\cos(\vartheta)) = (A_l r^l + B_l r^{-(l+1)})\underbrace{g_l(x)}_{P_l}$$

$$\phi(r,\vartheta,\varphi) = \sum_i P_i$$

$$\phi(r,\vartheta,\varphi) = \sum_{i=0}^{\infty}\sum_{m=-l}^{+l}(A_{lm}r^l + B_{lm}r^{-(l+1)})Y_{lm}(\vartheta,\varphi)$$

6.2 Beispiel: ungeladene Metallkugel

ungeladene Metallkugel in homogenen E-Feld $\vec{E}_0$



a) Symmetrie: Axialsymmetrie: keine φ-Abh.
   Allg. Potential: $\phi(r,\vartheta) = \sum_l (A_l r^l + B_l^{-(l+1)})(2l+1)P_l(\cos(\vartheta))$

b) Randbedingung:

   i) $r \to \infty \Rightarrow -\vec{\nabla}\phi^{(a)} = \vec{E}_0\vec{e}_z \Leftrightarrow -\int E_0 dz = \phi^{(a)} = -E_0 z$

   ii) $r < R,\ E_{(i)} = 0 \Leftrightarrow -\vec{\nabla}\phi^{(i)} = E^{(i)} \Rightarrow \phi^{(i)} = const = 0$

   iii) $\phi^{(i)}(R) = \phi^{(a)}(R)$

   aus $(ii) \Rightarrow A_l^{(i)} = B_l^{(i)} = 0, \forall l$

   $iii) \Rightarrow 0 = \phi^{(a)}(R) = \sum_i (A_l R^l + B_l R^{-(l+1)})P_l(\cos(\vartheta))(2l+1)$

   $A_l R^l = \frac{-B_l}{R^{l+1}} \Leftrightarrow B_l = -A_l R^{2l+1}$

   aus (i) $\Rightarrow \phi^a(r\to\infty) = -E_0 z = -E_0 r\cos(\vartheta) = -E_0 r P_1(\cos(\vartheta))$

   $\phi^{(a)}(r) = \sum_i (A_l r^l - A_l R^{2l+1} r^{-(l+1)})(2l+1)Pl(\cos(\vartheta))$

   $\phi^{(a)}(r\to\infty) = 3A_1 r P_1(\cos(\vartheta)) = -E_0 r P_1(\cos(\vartheta))$

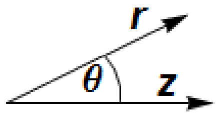
   $A_1 = -\frac{1}{3}E_0, \quad B_1 = E_0\frac{R^3}{3}$

$$\Rightarrow \phi(r,\vartheta) = \begin{cases} 0 & , r \le R \\ (-E_0 r + \frac{E_0 R^3}{r^2}P_1(\cos(\vartheta)) & , r > R \end{cases}$$

6.3 Kugel mit Oberflächen-Ladung

Kugel mit Oberflächenladung $\sigma(\theta) = \alpha(3\cos^2(\theta) - 1), \quad \alpha = const$

Ansatz: $\phi = \sum_{i=0}^{\infty}(2l+1)(A_l r^l + B_l r^{-(l+1)} P_l(\cos(\vartheta))$



i) Regularität im Ursprung: $r \to 0 : \phi^i(r=0) \to$ endlich

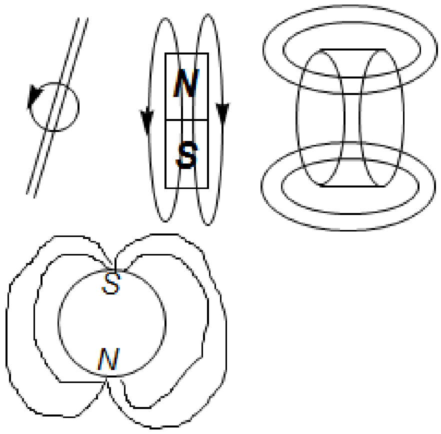
ii) Endlichkeit: $r \to \infty : \phi^a(r \to \infty) \to$ endlich

iii) $\phi(R) = \phi^a(R)$

iv $\sigma(\theta) = -\varepsilon_0(\frac{\partial \phi^a}{\partial r} - \frac{\partial \phi^i}{\partial r})|_{r=R}$

7 Elektrodynamik Tutorium vom 21.1.2011

7.1 Magnetfelder



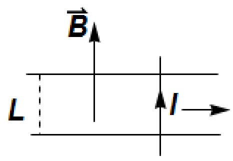
7.2 Induktionsgesetz

$U = \int \vec{\nabla} \times (\vec{\nabla} \times \vec{B}) d\vec{f} = -\int \Delta \vec{B} d\vec{F} ...$

$= -\frac{d}{dt} \int \vec{B} d\vec{F}$ (für $\frac{\partial \vec{B}}{\partial t} = 0$)

magnetischer Fluss: $\phi = \int \vec{B} d\vec{F} \to U = \frac{d}{dt}\phi$

7.3 Beispiel: Leiterschleife durch Magnetfeld



$\vec{F}_j = \int \vec{j} \times \vec{B} dV$

$x(t=0) = \dot{x}(t=0) = 0$

$I = \int_F \vec{j} dF$

$\vec{F}_L = \int \vec{j} \times \vec{B} dV = B \int \vec{j} \times \vec{e}_z dV$

$= B \int I \vec{e}_y \times \vec{e}_x dy$

$F = m\dot{v} = IBL \Rightarrow \dot{v} = \frac{IBL}{m}$

$v = \int \dot{v} dt = \dot{v}t + v_0$

$\Rightarrow v = \frac{IBL}{m} t$

$U_{ind} = -\frac{d}{dt} \int \vec{B} d\vec{F} = -\frac{d}{dt} \int B \vec{e}_z \vec{e}_z dF$

$= -B \frac{d}{dt} \int dF = -B \frac{d}{dt} L x(t) = -BLv = -\frac{IB^2L^2}{m} t$

7.4 Maxwell-Gleichungen

stationär:

$$\vec{\nabla}\vec{D} = \rho$$
$$\vec{\nabla} \times \vec{H} = \vec{j}$$
$$\vec{\nabla}\vec{B} = 0$$
$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

allgemein:

inhomogen	homogen
$\vec{\nabla}\vec{D} = \rho$	$\vec{\nabla}\vec{B} = 0$
$\vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{j}$	$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$
$\vec{B} = \mu\mu_0\vec{H}$	$\vec{D} = \varepsilon\varepsilon_0\vec{E}$

$$\Rightarrow \left.\begin{array}{l} \vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \\ \vec{B} = \vec{\nabla} \times \vec{A} \end{array}\right\} \begin{array}{l} \vec{\nabla}^2\vec{A} - \frac{1}{k^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu\mu_0\vec{j} \\ \vec{\nabla}^2\phi - \frac{1}{k^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\varepsilon\varepsilon_0} \end{array}, \quad k = \frac{1}{\sqrt{\mu\mu_0\varepsilon\varepsilon_0}} = c$$

$\Box = \vec{\nabla}^2 - \frac{1}{k^2}\frac{\partial^2}{\partial t^2}$ „Quabla“

7.5 em-Wellen im Vakuum

$$\varepsilon_0\vec{E} = \vec{D},\ \vec{B} = \mu_0\vec{H},\ \rho = 0,\ \vec{j} = 0$$
$$\varepsilon_0\vec{\nabla}\vec{E} = 0, \quad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0, \quad \vec{\nabla} \times \vec{B} = \varepsilon_0\mu_0\frac{\partial \vec{E}}{\partial t}, \quad \vec{\nabla}\vec{B} = 0$$

$$\Rightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \varepsilon_0\mu_0\frac{\partial}{\partial t}\underbrace{(\vec{\nabla} \times \vec{E})}_{-\frac{\partial \vec{B}}{\partial t}}$$
$$\Rightarrow -\Delta\vec{B} = -\varepsilon_0\mu_0\frac{\partial^2}{\partial t^2}\vec{B}$$
$$\Delta\vec{B} - \underbrace{\varepsilon_0\mu_0}_{\frac{1}{c^2}}\frac{\partial^2}{\partial t^2}\vec{B} = 0$$
$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B})$$
$$\Rightarrow \Delta E = -\underbrace{\varepsilon_0\mu_0}_{\frac{1}{c^2}}\frac{\partial^2}{\partial t^2}\vec{E}$$

7.6 Lösung d. Wellengleichung

$$\vec{E} = \vec{E}_0 e^{-\vec{k}\vec{r}-\omega t}$$
$$\Delta\vec{E} = \vec{E}_0\Delta e^{-i(\vec{k}\vec{r}-\omega t)} = -\vec{k}^2\vec{E}$$
$$\frac{\partial^2\vec{E}}{\partial t^2} = -\omega^2\vec{E}, \quad \omega^2 = \vec{k}^2c^2$$
$$(-k^2 + \frac{\omega^2}{c^2})\vec{E} = 0$$

7.7 Eigenschaften von em-Wellen

Wie stehen $\vec{k}, \vec{E}, \vec{B}$

$$\vec{\nabla}\vec{E} = 0 = i\vec{k}\vec{E}, \quad \vec{k}\perp\vec{E}\ \vec{\nabla}\vec{B} = 0 = i\vec{k}\vec{B}, \quad \vec{k}\perp\vec{B}\ \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\Rightarrow i(\vec{k} \times \vec{E}) = i\omega\vec{B}, \quad \vec{E}\perp\vec{B}$$

8 Elektrodynamik Tutorium vom 28.1.2011

8.1 Wellengleichung

homogene Wellengleichung: $\Box\psi(\vec{r},t)$

$\Box = \Delta - \frac{1}{u^2}\frac{\partial^2}{\partial t^2}$

$\psi(\vec{r},t) = f_-(\vec{k}\vec{r}-\omega t) + f_+(\vec{k}\vec{r}+\omega t), \quad u^2k^2 = \omega^2$

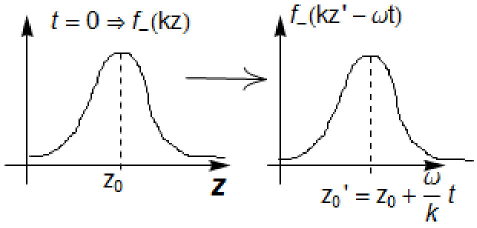
$\vec{k} = k\vec{e}_z \Rightarrow \vec{k}\vec{r} = kz$

$k = kz' - \omega t$

$z = z' - \frac{\omega}{k}$

$z_0' = z_0 + \frac{\omega}{k}t$

$\frac{dz'}{dt} = \frac{\omega}{k} = u$



Ansatz: $\quad f_-(\vec{r},t) = Ae^{i(\vec{k}\vec{r}-\omega k)}$

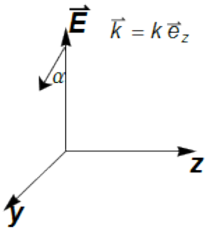
$f_+(\vec{r},t) = Be^{i(\vec{k}\vec{r}+\omega t)}$

$\Rightarrow \vec{E} = \vec{E}_0 e^{i(\vec{k}\vec{r}-\omega t)}, \quad \vec{B} = \vec{B}_0 e^{i(\vec{k}'\vec{r}-\omega' t)}$

Maxwell: $\vec{\nabla}\vec{E} = 0 \qquad \vec{\nabla}\times\vec{E} = -\dot{\vec{B}}$

$\vec{\nabla}\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \frac{1}{u^2}\dot{\vec{E}}$

$\Rightarrow \vec{k}\perp\vec{B},\vec{E} \qquad \vec{\nabla}\times\vec{E} = -\vec{B}$



$i(\vec{k}\times\vec{E}_0)e^{i(\vec{k}\vec{r}-\omega t)} = i\omega'\vec{B}_0 e^{i(\vec{k}'\vec{r}-\omega' t)}$

$k = k', \quad \omega = \omega' \quad \Rightarrow \vec{k}\times\vec{E}_0 = \omega\vec{B}_0, \quad \vec{k}\times\vec{B}_0 = \frac{\omega}{u^2}\vec{E}_0,$

$\vec{E}_0 = E_{0x}\vec{e}_x + E_{0y}\vec{e}_y$

$\vec{B}_0 = B_{0x}\vec{e}_x + B_{0y}\vec{e}_y$

$\vec{k}\times\vec{B}_0 = kB_{0x}\vec{e}_y - kB_{0y}\vec{e}_x$

$= -\frac{\omega}{u^2}e_{0x}\vec{e}_x - \frac{\omega}{u^2} - e_{0y}\vec{e}_y$

$B_{0x} = -\frac{\omega}{u^2k}E_{0y} = -\frac{\omega}{u}E_{0y}$

$B_{0y} = \frac{\omega}{u^2k}E_{0x} = \frac{\omega}{u}E_{0x}$

$\Rightarrow \vec{E} = (E_{0x}\vec{e}_x + E_{0y}\vec{e}_y)e^{i(kz-\omega t)}$

$\Rightarrow \vec{B} = (-\frac{1}{u}E_{0x}\vec{e}_x + \frac{1}{u}E_{0y}\vec{e}_y)e^{i(kz-\omega t)}$

$E_{0x} = |E_{0x}|e^{i\varphi}, \quad E_{0y} = |e_{0y}|e^{i\varphi+\delta}$

$\vec{E} = (|E_{0x}|\vec{e}_x + |E_{0y}|\vec{e}_y e^{i\delta})e^{i(kz-\omega t+\varphi}$

$\Re(\vec{E}) = |E_{0x}|\vec{e}_x\cos(\xi) + |E_{0y}|\vec{e}_y e^{i\delta}(\cos(\delta)\cos(\xi) - \sin(\delta)\sin(\xi))$

1. Fall: $\delta = 0, \pm\pi$

   $\Rightarrow \Re(\vec{E}) = (|E_{0x}|\vec{e}_x + |E_{0y}|\vec{e}_y)\cos(\xi)$

   $\tan(\alpha) = \frac{\pm|E_{0y}|}{|E_{0x}|}$

   (linear Polarisiert)

2. Fall $\delta = \pm\frac{\pi}{2}$

   $\Rightarrow \cos(\delta) = 0, \ \sin(\delta) = \pm 1$

   $\Rightarrow \Re(\vec{E}) = (|E_{0x}|\vec{e}_x\cos(\xi) \mp |E_{0y}|\vec{e}_y\sin(\xi))$

8.2 Fourier-Transformation

$f(x) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} dk\tilde{F}(k)e^{ikx}$

$\tilde{F}(x) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} dx\tilde{f}(k)e^{-ikx}$

$\vec{k}\vec{x} = k_x x + k_y y + k_z z \Rightarrow e^{i\vec{k}\vec{x}} = e^{ik_x x} + e^{ik_y y} + e^{ik_z z}$

3dim: $\frac{1}{\sqrt{2\pi}^3} \int dr^3 f(\vec{x}) e^{i\vec{k}\vec{x}}$

$\psi(\vec{r}, k) = \frac{1}{(2\pi)^2} \int d^3k \int d\omega \tilde{\psi}(\vec{k}, \omega) e^{i(\vec{k}\vec{r} - \omega t)}$

$\delta(\vec{k} - \vec{k}') = \frac{1}{(2\pi)^3} \int\limits_{-\infty}^{\infty} d^3 r e^{i\vec{x}(\vec{k} - \vec{k}')}$

$\Rightarrow \frac{1}{(2\pi)^2} \int\limits_{-\infty}^{\infty} d^3k \int\limits_{-\infty}^{\infty} d\omega (-k^2 + \frac{\omega^2}{u^2}) \tilde{\psi}(\vec{k}, \omega) e^{i(\vec{k}\vec{r} - \omega t} = 0$

Multiplizieren mit $e^{-i(\vec{k}'\vec{r} - \omega' t)} \int\limits_{-\infty}^{\infty} d^3 r \int\limits_{-\infty}^{\infty} dk$

$e^{i\vec{k}\vec{r}} e^{-i\omega t} e^{-i\vec{k}'\vec{r}} = e^{i(\vec{k} - \vec{k}')\vec{r}} e^{i(\omega' - \omega)t}$

$\Rightarrow (-k'^2 + \frac{\omega^2}{u^2}) \tilde{\psi}(\vec{k}', \omega') = 0$

9 Elektrodynamik Tutorium vom 11.2.2011

9.1 Aufgabe 1

Aufgabe:

$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$

$f(t) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \tilde{f}(\omega) e^{i\omega x} d\omega$

$f(t) = e^{-(\frac{t}{\Delta t})^2} \Rightarrow \tilde{f}(\omega) = ?$

Lösung:

$\tilde{f}(\omega) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} e^{-(\frac{t}{\Delta t})^2} e^{-i\omega t} dt$

$= \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} e^{-((\frac{t}{\Delta t})^2 + i\omega t)} dt$

$= \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} e^{-\frac{(t^2 + i\omega t \Delta t + (\frac{1}{2} i\omega \Delta t^2)^2) + (\overbrace{\frac{1}{2} i\omega\Delta t}^{const})^2}{\Delta t^2}} dt$

$= \frac{1}{\sqrt{2\pi}} e^{-\frac{\frac{1}{4}\omega^2 \Delta t^4}{\Delta t^2}} \int\limits_{-\infty}^{\infty} e^{-\frac{(t + \frac{1}{2} i\omega \Delta t^2)^2}{\Delta t^2}} dt$

Jetzt: Substitution $z = t + \frac{1}{2} i\omega t^2$, $\frac{\partial z}{\partial t} = 1$

$\Rightarrow ... = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{4}\omega^2 \Delta t^2} \int\limits_{-\infty}^{\infty} e^{-\frac{z^2}{\Delta t^2}} dz = \frac{1}{\sqrt{2}} \Delta t e^{-\frac{1}{4}\omega^2 \Delta t^2}$

Bemerkung: Fouriertransformation von Gaußfkt. ist immer Gaußfkt.

9.2 Aufgabe 2

$f(t) = e^{-\lambda t} \Rightarrow \tilde{f}(\omega) = ?$

$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^{-\lambda t} e^{-i\omega t} dt = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^{(-\lambda - i\omega)t} dt$

$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{-\lambda - i\omega} e^{(-\lambda - i\omega)t} \right]_0^{\infty} = \frac{1}{\sqrt{2\pi}} \frac{1}{\lambda + i\omega}$

9.3 Aufgabe 3: 3-Dim Fouriertransf.

Aufgabe:

$\tilde{f}(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \int d^3r f(\vec{r}) e^{i\vec{k}\vec{r}}$

mit $e^{i\vec{k}\vec{r}} = e^{ik_x x + ik_y y + ik_z z}$ und $\boxed{f(r) = \frac{e^{-\mu r}}{r}}$ $(\mu > 0)$

Hinweis: $\int\limits_0^\infty x^n e^{-\eta x} dx = n! \eta^{-n+1}$

Lösung:

$$\tilde{f}(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \int d^3r \frac{e^{-\mu r}}{r} e^{i\vec{k}\vec{r}}$$
$$= ... = \frac{1}{\sqrt{2\pi}} \int\limits_0^\infty \int\limits_0^\pi r e^{-\mu r} e^{ikr\cos(\theta)} d\cos(\theta) dr$$
$$= \frac{1}{\sqrt{2\pi}} \int\limits_0^\infty e^{-\mu r} \left[e^{-ikr} - e^{ikr}\right] dr$$
$$= \frac{i}{k\sqrt{2\pi}} \left(\int\limits_0^\infty e^{-r(\mu+ik)} dr - \int\limits_0^\infty e^{-r(\mu-ik)} dr \right)$$
$$= \frac{i}{k\sqrt{2\pi}} \left[\frac{1}{\mu+ik} - \frac{1}{\mu-ik} \right] = \frac{2}{\sqrt{2\pi}(\mu^2+k^2)}$$

9.4 linearer, homogener, aufgeladener Isolator

Maxwellgl: $\vec{\nabla}\vec{E} = 0$ $\quad \vec{\nabla}\vec{B} = 0$

$\vec{\nabla} \times \vec{E} = -\dot{\vec{B}}$ $\quad \vec{\nabla} \times \vec{B} = \frac{1}{u^2}\dot{\vec{E}}$

Aufgabe dazu:

$\vec{E}(\vec{r},t) = \frac{E_0}{5}(\vec{e}_x - 2\vec{e}_y) e^{i(\vec{k}\vec{r} - \omega t)}$ mit $\vec{k} = k\vec{e}_z$ $\vec{B}(\vec{r},t) = ?$

$$\Rightarrow \vec{\nabla} \times \vec{E} = -\dot{\vec{B}} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} x ...$$

Ergebnis: $\vec{B}(\vec{r},t) = \frac{E_0}{5}\frac{k}{\omega} e^{i(kz-\omega t)}(2\vec{e}_x + \vec{e}_y)$

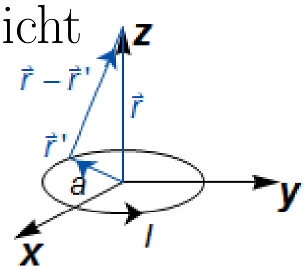
$\Rightarrow$ Linear polarisiert, da $2\vec{e}_x + \vec{e}_y$ nicht zeitabhängig

Aufgabe 2:

$$\vec{B}(\vec{r},t) = B_0 \cos(kz - \omega t)\vec{e}_x + B_0 \sin(kz - \omega t)\vec{e}_y$$
$$\vec{\nabla} \times \vec{B} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times B_0 \begin{pmatrix} \cos(kz-\omega t) \\ \sin(kz-\omega t) \\ 0 \end{pmatrix} = -B_0 k \begin{pmatrix} \cos(kz-\omega t) \\ \sin(kz-\omega t) \\ 0 \end{pmatrix}$$
$$\Rightarrow B_0 \frac{k}{\omega}\frac{1}{u^2} \begin{pmatrix} -\sin(kz-\omega t) \\ \cos(kz-\omega t) \\ 0 \end{pmatrix} = u B_0 \begin{pmatrix} -\sin(kz-\omega t) \\ \cos(kz-\omega t) \\ 0 \end{pmatrix}$$

10 Elektrodynamik Tutorium vom 18.2.2011

Klausurvorbereitung: Aufgabenübersicht



10.1 Biot-Savart

Biot-Savart: $B(\vec{r}) = \frac{\mu_0 I}{4\pi} \int\limits_c d\vec{r}' \times \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$

$$\vec{r}' = \vec{r}'(\varphi) = \begin{pmatrix} a\cos(\varphi) \\ a\sin(\varphi) \end{pmatrix}, \quad \varphi \in [0, 2\pi)$$

$$\vec{r}_0 ? \vec{r} - \vec{r}' = \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} - \begin{pmatrix} a\cos(\varphi) \\ a\sin(\varphi) \\ 0 \end{pmatrix} = \begin{pmatrix} -a\cos(\varphi) \\ -a\sin(\varphi) \\ z \end{pmatrix}$$

$$|\vec{r}_0^3| = (a^2 + z^2)^{3/2}$$

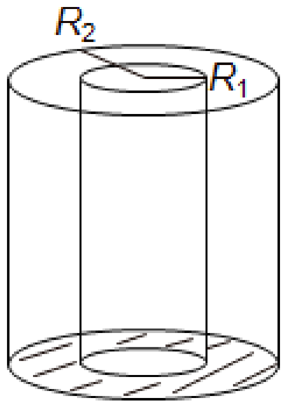
$$d\vec{r}' = \frac{d\vec{r}}{d\varphi} d\varphi = \begin{pmatrix} -a\sin(\varphi) \\ a\cos(\varphi) \end{pmatrix}$$

$$d\vec{r} \times \vec{r}_0 = \begin{pmatrix} az\cos(\varphi) \\ az\sin(\varphi) \\ a^2 \end{pmatrix} d\varphi$$

$$\Rightarrow \vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{1}{r_0^3} \begin{pmatrix} az\cos(\varphi) \\ az\sin(\varphi) \\ a^2 \end{pmatrix} d\varphi$$

$$= \frac{\mu_0 I}{2} \frac{a^2}{(a^2+z^2)^{3/2}}$$

10.2 Magnetfeld



Gefragt: B im ganzen Raum

$$\oint_c \vec{B} d\vec{r} = \mu_0 \int_{S(c)} \vec{j} d\vec{f}$$

$$\vec{B}(\vec{r}) = B(\vec{r}) \vec{e}_\varphi$$

L.S.: $\oint_c \vec{B} d\vec{r} = \int_0^{2\pi} rB(\vec{r}) d\varphi = 2\pi r B(\vec{r})$

R.S.:

1. Fall $r \leq R_1$:
   $B(\vec{r}) = 0$, da j für $r \leq R_1 0$ ist
2. Fall $R_1 \leq r \leq R_2$:
   $$\mu_0 \int_0^r \int_0^{2\pi} jrdrd\varphi$$
   $$= \mu_0 \left[\int_0^{R_1} \int_0^{2\pi} j_i rdrd\varphi + \int_{R_1}^r \int_0^{2\pi} rjdrd\varphi \right]$$
   $$= \mu_0 \pi j (r^2 - R_1^2)$$
   $$I = \pi j (R_2^2 - R_1^2) \Rightarrow j = \frac{I}{\pi(R_2^2 - R_1^2)}$$
   $$\Rightarrow ... = \mu_0 I \frac{(r^2 - R_1^2)}{(R_2^2 - R_1^2)}$$
3. Fall $R_2 \leq r$:
   $$\mu_0 \int_0^r \int_0^{2\pi} ... = \mu_0 \left[0 + \int_{R_1}^{R_2} \int_0^{2\pi} jrdrd\varphi + 0 \right]$$
   $$= \pi \mu_0 j (R_2^2 - R_1^2) = \mu_0 I$$

$$B(\vec{r}) = \frac{\mu_0 I}{2\pi} \begin{cases} 0 & r \leq R_1 \\ \frac{r}{R_2^2 - R_1^2} & R_1 \leq r \leq R_2 \\ \frac{1}{r} & R_2 \leq r \end{cases}$$

10.3 Schwingkreis

$$U_R = IR$$
$$U_C = \frac{U}{C}$$
$$U_L = -L\dot{I}$$
$$I = \dot{Q}$$
$$U_R = U_e - U_C + U_L \Leftrightarrow IR = U_e - \frac{Q}{C} - L\dot{I}$$

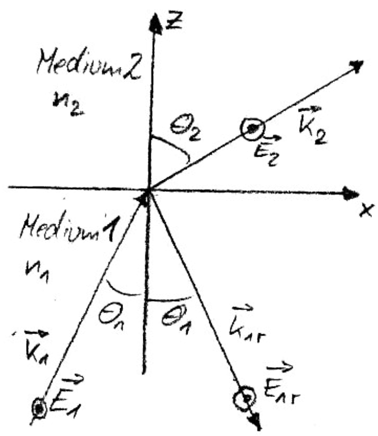
10.4 Wellenpakete

$$H_\pm = \int\limits_{-\infty}^{\infty} b(k)e^{i(kz\pm wt)}dt$$

10.5 Energie-/Energiestromdichte

$$\overline{A}(\vec{r}) = \frac{1}{t}\int\limits_{t}^{t+\tau} dt' A(\vec{r},t')$$
$$\omega(\vec{r},t) = \frac{1}{2}(\vec{H}(\vec{r},t)\vec{B}(\vec{r},t) + \vec{E}(\vec{r},t)\vec{D}(\vec{r},t))$$
$$\vec{s}(\vec{r},t) = (\vec{E}(\vec{r},t) \times \vec{H}(\vec{r},t))$$

10.6 Brechung/Reflexion



$$\vartheta_1 = \vartheta_{1r}$$
$$\frac{\sin(\vartheta_1)}{\sin(\vartheta_2)} = \frac{n_2}{n_1}$$
$$R = \left|\frac{\hat{n}-1}{\hat{n}+1}\right|^2 \text{ und } R + T = 1$$

10.7 Blatt 13, NR.3

1) $\vec{\nabla} \times [\vec{E}_2 - (\vec{E}_1 + \vec{E}_{1r})] = 0$
2) $\vec{\nabla} \cdot [\varepsilon_{r2}\vec{E}_2 - \varepsilon_{r1}(\vec{E}_1 + \vec{E}_{1r})] = 0$
3) $\vec{\nabla} \times [\frac{1}{\mu_{r2}}(\vec{k}_2 \times \vec{E}_2) - \frac{1}{\mu_{r1}}(\vec{k}_1 \times \vec{E}_1 + \vec{k}_{1r} \times \vec{E}_{1r}] = 0$
4) $\vec{\nabla} \cdot [(\vec{k}_2 \times \vec{E}_2) - (\vec{k}_1 \times \vec{E}_1 + \vec{k}_{1r} \times \vec{E}_{1r})] = 0$

$$\vec{k} = \begin{pmatrix} k_x \\ 0 \\ k_z \end{pmatrix} = \begin{pmatrix} k\sin(\theta) \\ 0 \\ k\cos(\theta) \end{pmatrix}$$
$$\vec{E} = \begin{pmatrix} 0 \\ E_y \\ 0 \end{pmatrix}$$

- $\vec{k}\vec{E} = 0$
- $\vec{E}\vec{e}_z = 0$
- $\vec{I} \times \vec{E} = (-E_y k\cos(\theta), 0, E_y k\sin(\theta))$
- $(\vec{k} \times \vec{E})\vec{e}_z = E_y k\sin(\theta)$
- $\vec{E} \times \vec{e}_z = (e_y, 0, 0)$
- $(\vec{k} \times \vec{E}) \times \vec{e}_z = (0, E_y k\cos(\theta), 0)$

3 $-E_{02}k_{z2} + E_{01}k_{z1} + E_{1r}k_{z1r} = 0$

① $-E_{02} + E_{01} + E_{01r} = 0 \Rightarrow E_{02} = E_{01} + E_{01r}$

$\Rightarrow -E_{01}k_{z2} - E_{01r}k_{2z} + E_{01}k_{z1} - E_{01r}k_{z1} = 0$

$E_{01}(k_{z1} - k_{z2}) = E_{01r}(k_{z2} + k_{z1})$

$\frac{E_{01r}}{E_{01}} = \frac{k_{z1}-k_{z2}}{k_{z2}-k_{z1}} = \frac{k_1 \cos(\theta_1) - k_2 \cos(\theta_2)}{k_2 \cos(\theta_2) + k_1 \cos(\theta_1)}$

$k_2 = k_1 \frac{n_2}{n_1}$

$\Rightarrow \frac{E_{01r}}{E_{01}} = \frac{k_1 \cos(\theta_1) - k_1 \frac{n_2}{n_1} \cos(\theta_2)}{k_1 \frac{n_2}{n_1} \cos(\theta_2) + k_1 \cos(\theta_1)}$

$= \frac{n_1 \cos(\theta_1) - n_2 \cos(\theta_2)}{n_2 \cos(\theta_2) + n_1 \cos(\theta_1)}$