Besprechung zu Blatt 10 zu Analysis 3

Aufgabe 1

$f_1(z) = \Re(z)$
$f_1(x+iy)0\Re(x+iy) = x \quad (x, y \in \mathbb{R})$
$\tilde{f_1} : \mathbb{R}^2 \to \mathbb{R}^2$
$(x, y) \mapsto (\Re(f_1(x+iy)), \Im(f_1(x+iy)))$
f_1 holomorph $\Leftrightarrow \tilde{f_1}$ rell diffbar.

$$J\tilde{f_1}(x_0, y_0) = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

$$\tilde{f_1}(x, y) = (x, 0) \Rightarrow J\tilde{f_1}(x_0, y_0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow f_1$ nicht holomorph.

$f_2(z) = \Im(z), f_2(x+iy) = y, \tilde{f_2}(x, y) = (y, 0)$

$$\Rightarrow J\tilde{f_2}(x_0, y_0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow f_2$ nicht holomorph

$$\lim_{w \to z_0} \frac{f(z_0) - f(w)}{z_0 - w} = \lim_{w \to z_0} \frac{z_0^2 - w^2}{z_0 - w}$$
$$= \lim_{w \to z_0} \frac{\cancel{(z_0 - w)}(z_0 + w)}{\cancel{z_0 - w}} = \lim_{w \to z_0} z_0 + w = 2z_0$$

$\Rightarrow f_3$ holomorph

$f_4(z) = |z|^2, f_4(x+iy) = |x+iy|^2 = \sqrt{x^2+y^2}^2 = x^2 + y^2$
$\tilde{f_4}(x, y) = (x^2 + y^2, 0)$

$$\Rightarrow J\tilde{f_4}(x_0, y_0) = \begin{pmatrix} 2x_0 & 2y_0 \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow$ holomorph bei 0, aber nicht auf $\mathbb{C} \setminus \{0\}$.

$f_5(x+iy) = g(x+iy) + ig(x+iy), \quad g : \mathbb{C} \to \mathbb{R}$ reell diffbar.
$\tilde{f_5}(x, y) = (g(x+iy), g(x+iy))$

$$J\tilde{f_5}(x_0, y_0) = \begin{pmatrix} \frac{\partial g(x_0+iy_0)}{\partial x} & \frac{\partial g(x_0+iy_0)}{\partial y} \\ \frac{\partial g(x_0+iy_0)}{\partial x} & \frac{\partial g(x_0+iy_0)}{\partial y} \end{pmatrix}$$

$\Leftrightarrow \frac{\partial g(x_0+iy_0)}{\partial x} = \frac{\partial g(x_0+iy_0)}{\partial y} \wedge -\frac{\partial g(x_0+iy_0)}{\partial y} = \frac{\partial g(x_0+iy_0)}{\partial x}$
$\Rightarrow \frac{\partial g(x_0+iy_0)}{\partial y} = -\frac{\partial g(x_0+iy_0)}{\partial y} \Rightarrow \frac{\partial g(x_0+iy_0)}{\partial y} = 0$
$\Rightarrow \frac{\partial g(x_0+iy_0)}{\partial x} = 0 \Rightarrow$ g konstant
$\Rightarrow f_5$ holomorph falls g konst.

$f_6(z) = z^2 + 2z + \overline{z}$
$f_6(x+iy) = (x+iy)^2 + 2(x+iy) + x - iy = (x^2 + 3x - y^2) + i(2xy + 2y - y)$
$\Rightarrow \tilde{f_6}(x, y) = (x^2 + 3x + y^2, 2xy + y)$

$$\Rightarrow J\tilde{f_6}(x0, y_0) = \begin{pmatrix} 2x_0 + 3 & -2y_0 \\ 2y_0 & 2x_0 + 1 \end{pmatrix}$$

$\Rightarrow f_6$ nicht holomprph

Aufgabe 2

$h : z \mapsto (f(z))^2$ holomorph $\Rightarrow f$ holomorph $g : z \mapsto z^2$ holomorph nach Aufgabe 1
$h(z) = (g \circ f)(z)$

$$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} = J\tilde{h}(x_0, y_0) = J\widetilde{(g \circ f)}(x_0, y_0) = J(\tilde{g} \circ \tilde{f})(x_0, y_0)$$

$$= J\tilde{g}(\tilde{f}(x_0,y_0)) \circ J\tilde{f}(x_0,y_0)$$
$$= \begin{pmatrix} \gamma & -\delta \\ \delta & \gamma \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a\gamma - c\delta & b\gamma - d\delta \\ a\delta + c\gamma & b\delta + d\gamma \end{pmatrix}$$
$$\Rightarrow a\gamma - c\delta = b\delta + d\gamma \ \wedge \ -b\gamma + d\delta = a\delta + c\gamma$$
$$\Rightarrow \underbrace{(a-d)}_{=0}\gamma + \underbrace{(-c-b)}_{=0}\delta = 0 \ \wedge \ \underbrace{(a-d)}_{=0}\delta + \underbrace{(c+b)}_{=0}\gamma = 0$$
$$\Rightarrow a - d = 0 \ \wedge \ c + b = 0$$
$$\Rightarrow a = d \ \wedge \ c = -b$$
$$\Rightarrow J\tilde{f}(x_0,y_0) = \begin{pmatrix} a & -c \\ c & a \end{pmatrix}$$
$$\Rightarrow f \text{ holomorph}$$

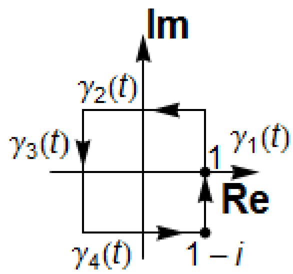
Aufgabe 3

Aufgabe 4

a) $\gamma(t) = e^{it}, \ t \in [0, 2\pi]$

$$\int_\gamma |z| dz = \int_0^{2\pi} |\gamma(t)|\dot{\gamma}(t)dt$$
$$= \int_0^{2\pi} i \underbrace{|e^{it}|}_{=1} e^{it} dt = \int_0^{2\pi} i^{it} dt = \int_0^{2\pi} i(\cos(t) + i\sin(t))dt$$
$$= \int_0^{2\pi} -\sin(t) + i\cos(t)dt = \int_0^{2\pi} -\sin(t)dt + i\int_0^{2\pi}\cos(t)dt = 0 + i0 = 0$$

b) $\gamma_1(t) = 1 - i + 2it, \ t \in [0,1]$

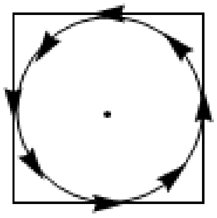


$$\gamma_2(t) = i\gamma_1(t)$$
$$\gamma_3(t) = i^2\gamma_1(t) = -\gamma_1(t)$$
$$\gamma_4(t) = i^3\gamma_1(t) = -i\gamma_1(t)$$
$$\gamma(t) = \begin{cases} \gamma_1(t) & , t \in [0,1] \\ \gamma_2(t-1) & , t \in [1,2] \\ \gamma_3(t-2) & , t \in [2,3] \\ \gamma_4(t-3) & , t \in [3,4] \end{cases}$$
$$\gamma : [0,4] \to \mathbb{C}$$
$$\int_\gamma \tfrac{1}{z} dz = \sum_{i=1}^{4} \int_{\gamma_i} \tfrac{1}{z} dz$$
$$\int_{\gamma_i} \tfrac{1}{z} dz = \int_0^1 \tfrac{1}{\gamma_i(t)} \dot{\gamma}_i(t)dt = \int_0^1 \tfrac{1}{\gamma_1(t)}\dot{\gamma}_1(t)dt$$

(da für i=2,3,4 sich der konst. Vorfaktor wegkürtzt)

$$\int_\gamma = 4\int_{\gamma_1} \tfrac{1}{z}dz = 4\int_0^1 \tfrac{1}{1-i+2it} \cdot (2i)dt$$
$$= 8i\int_0^1 \tfrac{1}{1+i(-1+2t)}dt = 8i\int_0^1 \tfrac{1-i(-1+2t)}{1^2+(-1+2t)^2}$$
$$= 8i\int_0^1 \tfrac{1+i(1-2t)}{1+(1-2t)^2}dt$$
$$= 8i\int_0^1 \tfrac{1}{1+(1-2t)^2}dt + 8i^2\int_0^1 \tfrac{1-2t}{1+(1-2t)^2}dt$$
$$= -4i\int_0^1 \tfrac{-2}{1+(1-2t)^2} + 2\int_0^1 \tfrac{-4(1-2t)}{1+(1-2t)^2}$$
$$= -4i[\arctan(1-2t)]_0^1 + 2[ln(1+(1-2t)^2)]_0^1$$

$$- 4i(\arctan(-1) - \arctan(1)) + 2\underbrace{(\ln(2) - \ln(2))}_{=0}$$
$$= -4i(-\arctan(1) - \arctan(1)) = 8i\arctan(1)$$
$$= 8i\tfrac{\pi}{4} = 2\pi i$$



oder:

$\gamma \equiv \tilde{\gamma},\ \tilde{\gamma}(t) = e^{2\pi i t},\ t \in [0,1]$

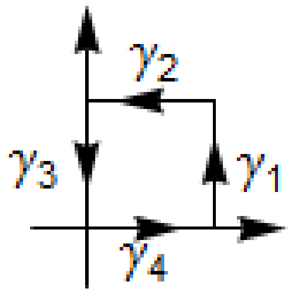
$$\int_\gamma \frac{1}{z}dz = \int_{\tilde{\gamma}} \frac{1}{z}dz = \int_0^1 \cancel{\frac{1}{e^{2\pi i t}}} 2\pi i \cancel{e^{2\pi i t}} dt$$

c) $\gamma_1(t) = 1 + it$

$\gamma_2(t) = 1 + i - t$

$\gamma_3(t) = i - it$

$\gamma_4(t) = t$



γ wie oben

$\int_\gamma |z|^2 dz$

(nicht holomorph! $\Rightarrow$ Homotopiesatz nicht anwendbar!)

$$\int_\gamma |z|^2 dz = \int_{\gamma_1} |z|^2 dz + ... + \int_{\gamma_4} |z|^2 dz$$

$$\int_{\gamma_1} |z|^2 dz = \int_0^1 |1+t|^2 i dt = \int_0^1 1^2 + t^2 i dt = i[t + \tfrac{1}{3}t^3]_0^1 = \tfrac{4}{3}i$$

$$\int_{\gamma_2} |z|^2 dz = \int_0^1 -|1-t+i|^2 dt = -\int_0^1 (1-t)^2 + 1 dt = -\tfrac{4}{3}$$

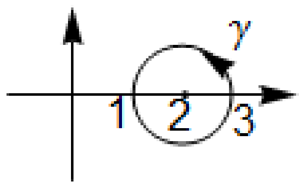
$$\int_{\gamma_3} |z|^2 dz = \int_0^1 -i(1-t)^2 dt = -\tfrac{1}{3}i$$

$$\int_{\gamma_4} |z|^2 dz = \int_0^1 t^2 dt = \tfrac{1}{3}$$

$$\Rightarrow \int_\gamma |z|^2 dz = \tfrac{4}{3}i - \tfrac{4}{3} - \tfrac{1}{3}i + \tfrac{1}{1} = i - 1$$

d) $\frac{1}{z}$ holomorph γ nullhomotop

$\Rightarrow$ mit Homot.Satz: $\int_\gamma \frac{1}{z}dz = 0$



Aufgabe 5

$$\int \underbrace{< i\overline{f}(\gamma(t)), \dot{\gamma}(t) >}_{\in \mathbb{R},\ =< \begin{pmatrix} v \\ u \end{pmatrix}, \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}} dt = i \underbrace{\int \underbrace{< \overline{f}(\gamma(t)), \dot{\gamma}(t) >}_{\in \mathbb{R}} dt}_{\in \mathbb{C} \setminus (\mathbb{R} \setminus \{0\})}$$