

Theo 4 Übungs-Tutorium

Mitgeschrieben und geL^AT_EXt von Julian Bergmann

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1 Tutorium vom 29.04.2011 (Blatt 1)

1.1 Hausaufgabe 1

Compton-Wellenlänge eines Teilchen mit Masse m : $\lambda_c = \frac{h}{mc}$

$$h = 6.626 \cdot 10^{-34} \text{ Js} = 4.1357 \cdot 10^{-15} \text{ eVs}$$

$$\hbar = \frac{h}{2\pi} = 1.0546 \cdot 10^{-34} \text{ Js} = 6.582 \cdot 10^{-16} \text{ eVs}$$

$$\hbar c = 197.33 \text{ MeV fm}$$

$$1 \frac{\text{MeV}}{c^2} = 1.79 \cdot 10^{-30} \text{ kg}$$

$$\text{a) } m_e = 9.1 \cdot 10^{-31} \text{ kg} = 0.51 \frac{\text{MeV}}{c^2}$$

$$\lambda_c(e^-) = \frac{2\pi\hbar}{m_e c} = \frac{2\pi\hbar c}{m_e c^2} = \frac{2 \cdot 3.14 \cdot 197.33 \text{ MeV fm}}{0.51 \text{ MeV}}$$

$$\approx 2429.86745 \text{ fm} \approx 2.43 \cdot 10^{-12} \text{ m}$$

b,c) analog

$$\text{d) } m(\text{Auto}) = t = 10^3 \text{ kg} = 0.56 \cdot 10^{33} \frac{\text{MeV}}{c^2}$$

$$\lambda_c(\text{Auto}) = \frac{2\pi\hbar}{m(\text{Auto})c} = \frac{2\pi\hbar c}{m c^2} = \frac{2 \cdot 3.14 \cdot 197.33 \text{ MeV fm}}{0.56 \cdot 10^{33} \text{ MeV}}$$

$$\approx 22.13 \cdot 10^{-31} \text{ fm} \approx 22.13 \cdot 10^{-46} \text{ m}$$

$$\lambda_{deBroglie} = \frac{h}{p} = \frac{h}{mv} \dots$$

1.2 Hausaufgabe 2

$\hbar = 10^{-3} \text{ Js}$ nur in Quantum-Land

$$d = 20 \text{ cm} = \Delta, \quad M = 0.1 \text{ g}$$

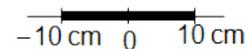
$$\boxed{\Delta x \cdot \Delta p \geq \frac{\hbar}{2}}$$

$$\Rightarrow \Delta P = \frac{\hbar}{2\pi} = \frac{10^{-3} \text{ kb} \frac{\text{m}}{\text{s}^2} \text{ ms}}{2 \cdot 0.2 \text{ m}} = 2.5 \cdot 10^{-3} \text{ kg} \frac{\text{m}}{\text{s}}$$

$$\Delta \sigma = \frac{\Delta p}{M} = \frac{2.5 \cdot 10^{-3} \text{ kg} \frac{\text{m}}{\text{s}}}{10^{-4} \text{ kg}} = 25 \frac{\text{m}}{\text{s}} = 90 \frac{\text{km}}{\text{h}}$$

$$\Delta v_{\min} \geq 90 \frac{\text{km}}{\text{h}}$$

d.h. Die Kerne besitzen min. $90 \frac{\text{km}}{\text{h}}$!



1.3 Hausaufgabe 3

a) **Lagrangefunktion:**

$$L = T - V = \frac{1}{2} m v^2 - \frac{e^2}{r}, \quad v^2 = \dot{r}^2$$

$$\vec{r}(t) = (r(t) \cos(\phi(t)), r(t) \sin(\phi(t)))$$

$$\dot{\vec{r}} = (\dot{r} \cos(\phi) - r \sin(\phi) \dot{\phi}, \dot{r} \sin(\phi) + r \cos(\phi) \dot{\phi})$$

$$\Rightarrow \dot{r}^2 = \dot{r}^2 + r^2 \dot{\phi}^2 = v^2$$

Hamiltonfunktion:

$$H = T + V = \frac{1}{2}mv^2 \frac{e^2}{r} = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 - \frac{e^2}{r}$$

Kanonische Größen:

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}, \quad p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

$$\Rightarrow \frac{p_r^2}{2m} = \frac{m^2\dot{r}^2}{2m} = \frac{1}{2}m\dot{r}^2, \quad \frac{p_\phi^2}{2m} = \frac{m^2r^4\dot{\phi}^2}{2mr^2} = \frac{1}{2}mr^2\dot{\phi}^2$$

$$H = T + V = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} - \frac{e^2}{r}$$

$$\frac{\partial L}{\partial \phi} = 0, \quad \frac{\partial L}{\partial \dot{\phi}} \neq 0 \Rightarrow \phi \text{ zyklisch.}$$

Konstanten der Bewegung:

(1) $H = E$, da $H(q, p)$ nicht explizit zeitabhängig.

(2) $p_\phi = L$, da ϕ zyklische Variable.

L: konstanter Drehimpuls.

Hamilton'sche Gleichungen:

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0, \quad \dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2}$$

$$\dot{p}_r = -\frac{\partial H}{\partial r} = -\frac{e^2}{r^2} + \frac{p_\phi^2}{mr^3}, \quad \dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}$$

b) Mit $p_\phi = L$:

$$E = \frac{p_r^2}{2m} + \frac{L^2}{2mr^2} - \frac{e^2}{r}$$

$$\Rightarrow p_r = \frac{\sqrt{2m(Er^2 + e^2r) - L^2}}{r}$$

c) Umkehrpunkte: $p_r = 0$

$$2m(Er^2 + e^2r) - L^2 = 0 \Rightarrow r^2 + \frac{e^2}{E}r - \frac{L^2}{2mE} = 0$$

$$\Rightarrow r_{1,2} = -\frac{e^2}{2E} \pm \frac{\sqrt{\Delta}}{4mE} \text{ mit } \Delta = -4m(2EL^2 + me^4)$$

d,e) wegen Einschüben fallen gelassen.

1.4 Einschub: harm. Oszillator

$$E = T + V = \frac{mq^2}{2} + \frac{fq^2}{2} = \frac{p^2}{2m} + \frac{fq^2}{2}$$

$$\frac{p^2}{2mE} + \frac{q^2}{2mE} = 1, \quad a = \sqrt{2mE}, \quad b = \sqrt{\frac{2E}{f}}$$

$$\frac{p^2}{a^2} + \frac{q^2}{b^2} = 1 \Rightarrow \text{Ellipse}$$

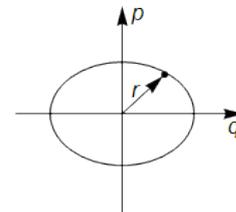
$$F = \int f(x)dx, \quad \text{Ellipse: } \oint pdq = \pi ab = \pi\sqrt{2mE}\sqrt{\frac{2E}{f}} = 2\pi E\sqrt{\frac{m}{f}}$$

$$\nu = \frac{1}{2\pi}\sqrt{\frac{f}{m}} \Rightarrow \oint pdq = \pi ab = 2\pi E\sqrt{\frac{m}{f}} = \frac{E}{\nu} = nh$$

Oszillator-Zustände: $E_n = nh\nu$

$$\Rightarrow \boxed{\oint pdq = nh}$$

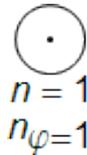
Hinweis zur Schreibweise: f=D=k



1.5 Bohr'sche Bahnen

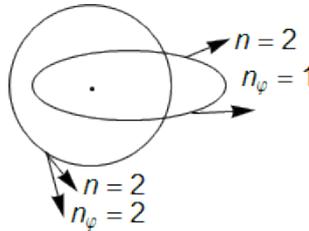
$$n = n_r + n_\varphi, \quad n_\varphi = 1, \dots, n$$

Bahnen: a)

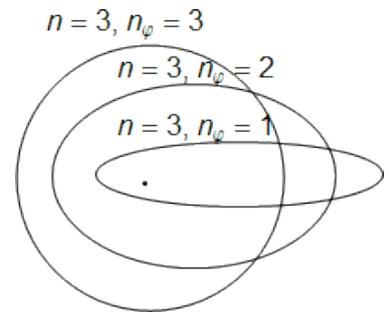


$$E_n = -\frac{me^4}{2n^2\hbar^2}$$

b)



c)



2 Tutorium vom 06.05.11 (Blatt 2)

2.1 Hausaufgabe 3

$$\ddot{x} + \omega_0^2 \sin(x) = F \cos(\omega t)$$

$$\begin{aligned} \text{a) } \sin(x) &= \sum_k (-1)^k \frac{x^{2k+1}}{(2k+1)!} \approx x - \frac{1}{6}x^3 \\ \Rightarrow \ddot{x} + \omega_0 x - \frac{1}{6}\omega_0 x^3 &= F \cos(\omega t) \end{aligned}$$

$$\begin{aligned} \text{b) } x' &= \frac{dx}{d\tau}, \quad \dot{x} = \frac{dx}{dt} = \frac{dx}{d\tau} \frac{d\tau}{dt} = x' \omega \\ \Rightarrow \ddot{x} &= \omega^2 x'' \\ \Rightarrow x'' + \Omega^2 x - \frac{1}{6}\Omega^2 x^3 &= \Gamma \cos(\tau) \end{aligned}$$

$$\begin{aligned} \text{c) } (x_0'' + \varepsilon_1'' + \dots) + \Omega^2(x_0 + \varepsilon_1 + \dots) - \varepsilon(x_0 + \varepsilon_1 + \dots)^3 &= \Gamma \cos(\tau) \\ x_0'' + \Omega^2 x_0 &= \Gamma \cos(\tau) \\ \Rightarrow \text{DGLs: } x_1'' + \Omega^2 x_1 &= x_0^3 \\ x_2'' + \Omega^2 x_2 &= 3x_0^2 x_1 \end{aligned}$$

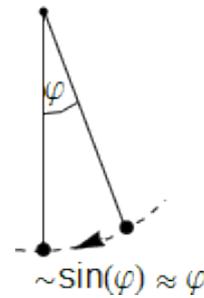
$$\text{d) } 2\pi\text{-periodisch nur f\u00fcr } a_0 = b_0 = 0$$

$$\begin{aligned} \text{e) } x_0(\tau) &= \frac{\Gamma}{\Omega^2 - 1} \cos(\tau) \text{ einsetzen.} \\ x_1'' + \Omega^2 x_1 &= \frac{\Gamma^3}{(\Omega^2 - 1)^3} \underbrace{\left(\frac{3}{4} \cos(\tau) + \frac{1}{4} \cos(3\tau) \right)}_{\cos^3(\tau)} \end{aligned}$$

$$\text{f) } x_1(\tau) = A \cos(\tau) + B \cos(3\tau) \Rightarrow x_1''(\tau) = -A \cos(\tau) - 9B \cos(3\tau)$$

$$\begin{aligned} x_1'' + \Omega^2 x_1 &= A(\Omega^2 - 1) \cos(\tau) + B(\Omega^2 - 9) \cos(3\tau) \stackrel{!}{=} \frac{3}{4} \frac{\Gamma^3}{(\Omega^2 - 1)^3} \cos(\tau) + \frac{1}{4} \frac{\Gamma^3}{(\Omega^2 - 1)^3} \cos(3\tau) \\ \Rightarrow A &= \frac{3}{4} \frac{\Gamma^3}{(\Omega^2 - 1)^4}, \quad B = \frac{1}{4} \frac{\Gamma^3}{(\Omega^2 - 1)^3 (\Omega^2 - 9)} \end{aligned}$$

$$\text{g) } x(\varepsilon, \tau) = \frac{\Gamma}{\Omega^2 - 1} \cos(\tau) + \varepsilon \left(\frac{3}{4} \frac{\Gamma^3}{(\Omega^2 - 1)^4} \cos(\tau) + \frac{1}{4} \frac{\Gamma^3}{(\Omega^2 - 1)^3 (\Omega^2 - 9)} \cos(3\tau) \right) + O(\varepsilon^2)$$



2.2 Hausaufgabe 4

$$\begin{aligned} \text{a) } \Psi(x, 0) &= \int_{-\infty}^{\infty} f(p) \Psi_p(x - x_0, 0) dp \\ &= \int_{-\infty}^{\infty} \underbrace{\frac{1}{(2\pi)^{1/4} \sqrt{\sigma_p}}}_{c_1} e^{-\frac{(p-p_0)^2}{4\sigma_p^2}} \underbrace{\frac{1}{(2\pi\hbar)^{1/2}}}_{c_2} e^{-\frac{i}{\hbar}(0-p(x-x_0))} dp \end{aligned}$$

Integrand:

$$\begin{aligned} e^{-\frac{(p-p_0)^2}{4\sigma_p^2}} e^{\frac{i}{\hbar}p(x-x_0)} &= e^{-\frac{p^2-2pp_0+p_0^2}{4\sigma_p^2}} e^{\frac{i}{\hbar}p(x-x_0)} \\ &= e^{-\frac{p^2}{4\sigma_p^2} + \frac{2p_0p}{4\sigma_p^2} + \frac{i}{\hbar}(x-x_0)p - \frac{p_0^2}{4\sigma_p^2}} \\ &= \exp\left(\underbrace{-\frac{1}{4\sigma_p^2}p^2}_a + \underbrace{\left[\frac{p_0}{2\sigma_p^2} + \frac{i}{\hbar}(x-x_0)\right]p}_b + \underbrace{\left(-\frac{p_0^2}{4\sigma_p^2}\right)}_c\right) \\ &\Rightarrow ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + c \\ \Psi(x, 0) &= c_1 c_2 \int_{-\infty}^{\infty} e^{a(p+\frac{b}{2a})^2} e^{c-\frac{b^2}{4a}} dp \\ &= c_1 c_2 e^{c-\frac{b^2}{4a}} \underbrace{\int_{-\infty}^{\infty} e^{-\alpha(p-\beta)^2} dp}_{=\sqrt{\frac{\pi}{\alpha}}}, \quad \text{mit } \alpha = \frac{1}{4\sigma_p^2} \\ &= c_1 c_2 e^{c-\frac{b^2}{4a}} \cdot 2\sigma_p \sqrt{\pi} \end{aligned}$$

Exponent:

$$\begin{aligned} x - \frac{b^2}{4a} &= -\frac{p_0^2}{4\sigma_p^2} - \frac{1}{4\left(-\frac{1}{4\sigma_p}\right)} \left[\frac{p_0}{2\sigma_p^2} + \frac{i}{\hbar}(x-x_0)\right]^2 \\ &= -\frac{p_0^2}{4\sigma_p^2} + \sigma_p^2 \left[\frac{p_0^2}{4\sigma_p^4} + i\frac{p_0(x-x_0)}{\sigma_p^2\hbar} - \frac{(x-x_0)^2}{\hbar}\right] \\ &= -\frac{(x-x_0)^2\sigma_p^2}{\hbar^2} + \frac{i}{\hbar}p_0(x-x_0) \end{aligned}$$

$$\text{Damit: } \Psi(x, 0) = \underbrace{\frac{1}{(2\pi)^{1/4} \sqrt{\sigma_p}} \cdot \frac{1}{(2\pi\hbar)^2} \cdot 2\sigma_p \sqrt{\pi}}_{\frac{1}{(2\pi)^{1/4} \sqrt{\frac{2\sigma_p}{\hbar}}}} e^{-\frac{\sigma_p^2}{\hbar^2}(x-x_0)^2} \cdot e^{\frac{i}{\hbar}p_0(x-x_0)}$$

$$\text{Für } t=0: \sigma_p \sigma_x = \frac{\hbar}{2} \begin{cases} \Rightarrow \sqrt{\frac{2\sigma_p}{\hbar}} = \frac{1}{\sqrt{\sigma_x}} \\ \Rightarrow \frac{\sigma_p^2}{\hbar^2} = \frac{1}{4\sigma_x^2} \end{cases}$$

$$\Rightarrow \Psi(x, 0) = \frac{1}{(2\pi)^{1/4} \sqrt{\sigma_x}} e^{-\frac{(x-x_0)^2}{4\sigma_x}} e^{\frac{i}{\hbar}p_0(x-x_0)} = M(x, 0) e^{i\phi(x, 0)}$$

$$\begin{aligned} \text{b) } \int_{-\infty}^{\infty} |\Psi(x, 0)|^2 dx &= \frac{1}{\sqrt{2\pi}\sigma_x} \int_{-\infty}^{\infty} e^{-\frac{(x-x_0)^2}{2\sigma_x}} dx \\ &= \frac{1}{\sqrt{2\pi}\sigma_x} \sqrt{\frac{\pi}{\frac{1}{2\sigma_x^2}}} = \frac{1}{\sqrt{2\pi}\sigma_x} \cdot \sqrt{2\pi}\sigma_x = 1 \end{aligned}$$

3 Tutorium vom 13.05.11 (Blatt 3)

3.1 Hausaufgabe 6

$$\Psi(x) = \mathcal{N} \exp\left(-\frac{(x-x_0)^2}{2\sigma^2}\right)$$

a) Substitution: $x - x_0 \rightarrow x$

$$\int_{-\infty}^{\infty} \Psi^2(x) dx = \mathcal{N}^2 \int_{-\infty}^{\infty} e^{-\frac{x^2}{\sigma^2}} dx = \mathcal{N} \sqrt{\pi} \sigma$$

$$\Rightarrow \mathcal{N} = (\pi \sigma^2)^{-\frac{1}{4}}$$

b) Nach Vorliebe von Herrn Prof. Dr. Lenske:

$$\Psi(p) = \int_{-\infty}^{\infty} e^{+ipx} \Psi(x) dx$$

$$\Psi(x) = \int_{-\infty}^{\infty} e^{-ipx} \Psi(p) \frac{dp}{2\pi}$$

$$\int_{-\infty}^{\infty} e^{-\alpha(x-\beta)^2} dx = \sqrt{\frac{\pi}{\alpha}}$$

$$\Psi(x) = \mathcal{N} \int_{-\infty}^{\infty} dx e^{ipx - \frac{(x-x_0)^2}{2\sigma^2}} = \mathcal{N} \int_{-\infty}^{\infty}$$

$$a = -\frac{(x-x_0)^2}{2\sigma^2} + ipx = -\frac{x^2}{2\sigma^2} + \frac{x_0 x}{\sigma^2} - \frac{x_0^2}{2\sigma^2} + ipx$$

$$= -\frac{1}{2\sigma^2} (x^2 - (2x_0 + ip \cdot 2\sigma^2)x + x_0^2)$$

$$= -\frac{1}{2\sigma^2} \left(\underbrace{(x + (x_0 + ip\sigma^2))^2}_{\tilde{x}} - (x_0 + ip\sigma^2)^2 + x_0^2 \right)$$

$$= -\frac{1}{2\sigma^2} (\tilde{x}^2 - 2ip\sigma x_0 + p^2\sigma^4)$$

Das Gauß-Integral über \tilde{x} liefert den Faktor $\sqrt{2\pi}\sigma$ und man erhält:

$$\Psi(p) = (4\pi\sigma^2)^{\frac{1}{4}} e^{-\frac{\sigma^2 p^2}{2} + ipx_0}$$

Die Fourier-Transformation ist norm-erhaltend: $\int_{-\infty}^{\infty} \frac{dp}{2\pi} |\Psi(p)|^2 = 1$

c) $x - x_0 \rightarrow u$ $n=1$:

$$\langle x \rangle = \int_{-\infty}^{\infty} dx \Psi^*(x) x \Psi(x) = \mathcal{N} \int_{-\infty}^{\infty} du (u + x_0) e^{-\frac{u^2}{\sigma^2}}$$

$$= \frac{1}{\sqrt{\pi}\sigma} = x_0 \sqrt{\sigma} = x_0, \text{ weil } \int_{-\infty}^{\infty} dx x e^{-\frac{x^2}{\sigma^2}} = 0$$

$$\langle p \rangle = 0$$

n=2:

$$\begin{aligned}
\langle x^2 \rangle &= \int_{-\infty}^{\infty} dx \Psi^*(x) x^2 \Psi(x) = \mathcal{N}^2 \int_{-\infty}^{\infty} dx' \underbrace{(x+x_0)^2}_{x^2+2x_0x+x_0^2} e^{-\frac{x'^2}{\sigma^2}} \\
&= \mathcal{N}^2 \left(\int_{-\infty}^{\infty} dx x^2 e^{-\frac{x^2}{\sigma^2}} + 2x_0 \underbrace{\int_{-\infty}^{\infty} dx x e^{-\frac{x^2}{\sigma^2}}}_{=0, \text{ ungerd. Integrant}} + \int_{-\infty}^{\infty} dx x_0^2 e^{-\frac{x^2}{\sigma^2}} \right) \\
&= \frac{1}{\sqrt{\pi}\sigma} \left(\frac{1}{2} \sqrt{\pi} \sigma \sqrt{\sigma} + x_0^2 \sqrt{\pi} \sigma \right) = \frac{1}{2} \sqrt{\sigma} + x_0^2 \\
\Psi(p) &= (4\pi\sigma^2)^{\frac{1}{4}} e^{-\frac{\sigma^2 p^2}{2} + ipx_0}
\end{aligned}$$

$$\begin{aligned}
\langle p^2 \rangle &= \int_{-\infty}^{\infty} \frac{dp}{2\pi} \Psi^*(p) p^2 \Psi(p) \\
&= (4\pi\sigma^2) \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{-\sigma^2 p^2} p^2 \\
&= 2\sigma \sqrt{\pi} i \frac{1}{2\pi} \frac{\sqrt{\pi}}{2(\sigma^2)^{\frac{3}{2}}} = \frac{1}{2\sqrt{\sigma}}
\end{aligned}$$

$$e) (\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{1}{2} \sqrt{\sigma} + x_0^2 - x_0^2 = \frac{1}{2} \sqrt{\sigma}$$

$$(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2 = \frac{1}{2\sqrt{\sigma}}$$

$$\Delta x \cdot \Delta p = \left(\frac{1}{2\sqrt{\sigma}} \cdot \frac{1}{2} \sqrt{\sigma} \right)^{\frac{1}{2}} = \frac{1}{2}$$

$$\Rightarrow \Delta x \Delta p \geq \frac{\hbar}{2} \text{ mit } \hbar = 1$$

$$d) 1. n=1: [x, p] = i\hbar, \quad [A, BC] = [A, B]C + B[A, C]$$

$$[x, p^n] = i\hbar n p^{n-1}$$

$$[x, p^{n+1}] = [x, p^n]p + p^n[x, p] = i\hbar n p^n + i\hbar p^n = i\hbar(n+1)p^{(n+1)-1}$$

$$2. [p, f(x)] = \alpha f'(x)$$

$$\text{Taylor: } f(x) = \sum_n \frac{1}{n!} f^{(n)}(0) x^n, \quad f'(x) = \sum_n \frac{1}{n!} (n f^{(n)}(0)) x^{n-1}$$

$$[p, f(x)] = \sum_n \frac{1}{n!} f^{(n)}(0) [p, x^n] = \sum_n \frac{1}{n!} f^{(n)}(0) (-i\hbar n x^{n-1}) = -i\hbar f'(x)$$

Auch über Anwendung auf $\Psi(x)$ und Ableitung.

3.2 Zusatzaufgabe

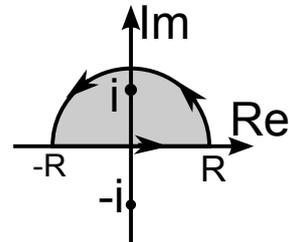
$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \arctan \Big|_{-\infty}^{\infty} = \pi$$

Residuen-Satz:

$$\int_{-\infty}^{\infty} \frac{dz}{z^2+1}$$

Nennernullstellen: $z = \pm i$

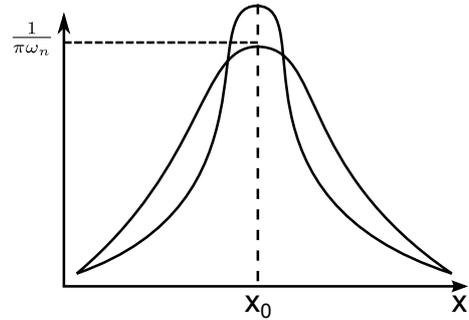
$$= 2\pi i \sum_{\alpha} \text{Res}_{\alpha} f(z) = 2\pi i \cdot \text{Res}_{z=i} f = 2\pi i \frac{g(z)}{h'(u)} \Big|_{z=i} = 2\pi i \frac{1}{2i} = \pi$$



3.3 Hausaufgabe 7

$$\ln(x) = A \frac{1}{(x-x_0)^2 + \omega_n^2}, \quad \omega_n = \frac{\omega_0}{n}, \quad n \in \mathbb{N}$$

$$\begin{aligned} \text{a) } A \int_{-\infty}^{\infty} \frac{1}{(x-x_0)^2 + \omega_n^2} dx &= \frac{A}{\omega_n^2} \int_{-\infty}^{\infty} \frac{1}{\left(\frac{(x-x_0)^2}{\omega_n^2}\right) + 1} \\ \text{Substitution: } \left(\frac{x-x_0}{\omega_n}\right)^2 &= y^2 \Rightarrow dy = \frac{dx}{\omega_n} \\ \frac{A}{\omega_n^2} \int_{-\infty}^{\infty} \frac{1}{1+y^2} \omega_n dy &= \frac{A}{\omega_n} \arctan(y) \Big|_{-\infty}^{\infty} = \frac{\pi A}{\omega_n} \stackrel{!}{=} 1 \\ \Rightarrow A &= \frac{\omega_n}{\pi} \end{aligned}$$



4 Tutorium vom 20.05.11 (Blatt 4)

4.1 Hausaufgabe 8

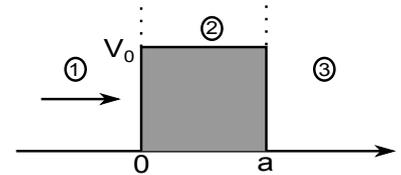
$$\text{a) } E < V_0, k^2 = \frac{2m}{\hbar^2} E, \quad \kappa = \frac{2m}{\hbar^2} (V_0 - E)$$

$$\psi_I(x) = A_1 e^{ikx} + B_1 e^{-ikx} \quad \text{für 1}$$

$$\psi_2(x) = A_2 e^{-\kappa x} + B_2 e^{\kappa x} \quad \text{für 2}$$

$$\psi_3(x) = A_3 e^{ikx} \quad \text{für 3}$$

($B_3 = 0$ da Teilchen von links ankommt, also kein Wellenteilchen von rechts.)



Anschlussbedingungen:

$$\psi_1(0) = \psi_2(0); \quad \psi_2(a) = \psi_3(a)$$

$$\psi_1'(0) = \psi_2'(0); \quad \psi_2'(a) = \psi_3'(a)$$

$$\psi_1'(x) = A_1 i k e^{ikx} - B_1 i k e^{-ikx}$$

$$\psi_2'(x) = -A_2 \kappa e^{-\kappa x} - B_2 \kappa e^{\kappa x}$$

$$\psi_3'(x) = A_3 i k e^{ikx}$$

$$(1) : A_1 + B_1 = A_2 + B_2$$

$$(2) : A_2 e^{-\kappa a} + B_2 e^{\kappa a} = A_3 e^{ika}$$

$$(3) : A_1 i k - B_1 i k = -A_2 \kappa + B_2 \kappa$$

$$(4) : -A_2 \kappa e^{-\kappa a} + B_2 \kappa e^{\kappa a} = A_3 i k e^{ika}$$

$$(4) \Rightarrow B_2 e^{\kappa a} = \frac{A_3 k e^{ika} + A_2 \kappa e^{-\kappa a}}{\kappa}$$

$$\text{Einsetzen in (2): } A_2 e^{-\kappa a} + \frac{A_3 k e^{ika} + A_2 \kappa e^{-\kappa a}}{\kappa} = A_3 e^{ika}$$

$$\Rightarrow A_2 (e^{-\kappa a} + e^{-\kappa a}) = A_3 (e^{ika} - \frac{ik}{\kappa} e^{ika})$$

$$\Rightarrow A_2 = \frac{e^{ika} - ik e^{ika}/\kappa}{2e^{-\kappa a}} A_3 = \boxed{A_2 = \frac{1}{2} e^{ika} e^{\kappa a} \left(1 - \frac{1}{\lambda}\right) A_3} \quad (\lambda = \frac{\kappa}{k})$$

A_2 einsetzen in (2):

$$B_2 = \frac{1}{2} e^{ika} e^{-\kappa a} \left(1 + \frac{1}{\lambda}\right) A_3$$

$$B_1 = -\frac{i}{2} e^{ika} \left(\lambda + \frac{1}{\lambda}\right) \sinh(\kappa a) A_3$$

$$(1) \Rightarrow A_1 = A_2 + B_2 - B_1$$

Für A_2, B_2 und B_1 einsetzen

$$\Rightarrow A_1 = e^{ika} \left(\cosh(\kappa a) + \frac{i}{2} \left(\lambda - \frac{1}{\lambda}\right) \sinh(\kappa a) \right) A_3$$

$$b) T = \left| \frac{A_3}{A_1} \right|^2, \quad R = \left| \frac{B_1}{A_1} \right|^2$$

$$T = \frac{1}{1 + \frac{1}{4} \left(\lambda + \frac{1}{\lambda}\right)^2 \sinh^2(\kappa a)}$$

$$R = \frac{\frac{1}{4} \left(\lambda + \frac{1}{\lambda}\right)^2 \sinh^2(\kappa a)}{1 + \frac{1}{4} \left(\lambda + \frac{1}{\lambda}\right)^2 \sinh^2(\kappa a)}$$

$$T + R = 1$$

$$c) \text{ klassische Mechanik: für } E < V_0 : T = 0, R = 1$$

Sehr breite Barriere: $\kappa a \gg 1 : \lim_{x \rightarrow \infty} \sinh(x) = \infty$

$$\Rightarrow T \rightarrow 0 \text{ für } \kappa a \rightarrow \infty$$

$$d) \text{ klassische Mechanik für } E \geq V_0 : T = 1, R = 0$$

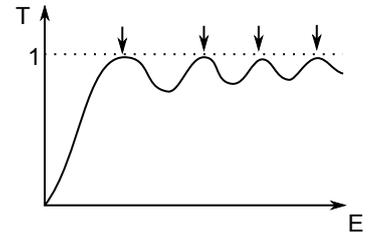
Für $E > V_0$ wird κ imaginär

$$\text{Setze daher } \kappa = -iK, \quad \lambda = \frac{\kappa}{K} = -i \frac{K}{k} = -i\Lambda$$

$$\text{Gl lauten dann: } T = \frac{1}{1 + \frac{1}{4} \left(\Lambda - \frac{1}{\Lambda}\right)^2 \sin^2(Ka)}$$

$$R = \frac{\frac{1}{4} \left(\Lambda - \frac{1}{\Lambda}\right)^2 \sin^2(Ka)}{1 + \frac{1}{4} \left(\Lambda - \frac{1}{\Lambda}\right)^2 \sin^2(Ka)}$$

$$e) \text{ Transmission wir 1, wenn } \sin(Ka) = 0, \text{ d.h. } \boxed{Ka = n\pi}$$



4.2 Hausaufgabe 9

$$a) V(x) = \begin{cases} \infty & , x < -a \\ 0 & , -a \leq x < 0 \\ V_0 \delta(x) & , x = 0 \\ 0 & , 0 < x \leq a \\ \infty & , x > a \end{cases}$$

Potential symm. um 0: Lösungen haben definierte Parität

$$b) \left. \begin{array}{l} \text{Region 1: } -a < x < 0 \\ \text{Region 2: } 0 < x < a \end{array} \right\} \text{ jeweils } V(x) = 0$$

$$-\frac{\hbar^2}{2m} \frac{d^2 \Psi}{dx^2} = E \Psi$$

$$\frac{d^2 \Psi}{dx^2} = -k^2 \Psi \text{ mit } k^2 = \frac{2mE}{\hbar^2}$$

c) Ansätze:

$$\psi_1(x) = A_1 \sin(kx) + B_1 \cos(ka) \psi_2(x) = A_2 \sin(kx) + B_2 \cos(ka)$$

Paritätsbed.:

$$\text{gerade: } \psi(-x) = \psi(x) \Rightarrow \psi_1(-x) = \psi_2(x)$$

$$\forall -A_1 \sin(kx) + B_1 \cos(kx) = A_2 \sin(kx) + B_2 \cos(kx) \Rightarrow B_1 = B_2, \quad A_1 = -A_2$$

$$\text{ungerade: } \Rightarrow A_1 = A_2, \quad B_1 = -B_2$$

$$\text{Stetigkeit bei } x = 0: \quad \psi_1(0^-) = \psi_2(0^+)$$

$$A_1 \sin(0) + B_1 \cos(0) = A_2 \sin(0) + B_2 \cos(0) \Rightarrow B_1 = B_2$$

$$\text{Für ungerade Parität gilt auch } B_1 = -B_2 \Rightarrow B_1 = B_2 = 0$$

Unstetigkeitssprung: der ersten Ableitung:

$$\text{Es gilt } \frac{-\hbar^2}{2m}(\psi_2'(0^+) - \psi_1'(0^-)) + V_0 \Psi(0) = 0$$

$$\psi_1'(x) = A_1 k \cos(kx) - B_1 k \sin(kx)$$

$$\psi_2'(x) = A_2 k \cos(kx) - B_2 k \sin(kx)$$

$$\varepsilon \rightarrow 0^+ :$$

$$\frac{-\hbar^2}{2m}(A_2 k \cos(k\varepsilon) - B_2 k \sin(k\varepsilon)) - (A_1 k \cos(k\varepsilon) - B_1 k \sin(k\varepsilon)) + V_0 B_1 = 0$$

$$\Rightarrow \frac{-\hbar^2}{2m}(A_2 k - A_1 k) + V_0 B_1 = 0 \quad (\text{für gerade Parität})$$

$$\text{gerade Par.: } A_1 = -A_2 \Rightarrow -\frac{\hbar^2}{2m}2A_2 k + V_0 B_1 = 0 \Rightarrow A_2 = \frac{mV_0}{\hbar^2 k} B_1$$

$$\psi_1(-a) = \psi_2(a) = 0$$

Fall „ungerade“: Mit $A_1 = A_2 = A$ haben wir den Ansatz

$$\psi_1(x) = A \sin(kx) = \psi_2(x)$$

$$\psi_2(a) = A \sin(ka), \quad A \neq 0 \Rightarrow k = \frac{n\pi}{a}$$

$$\sqrt{\frac{2mE}{\hbar^2}} \Rightarrow E_n = \frac{\pi^2 \hbar^2}{2ma^2} n^2$$

$$\int_{-\infty}^{\infty} |\Psi(x)|^2 dx = 1 \Rightarrow \int_{-a}^a A^2 \sin^2(kx) dx = A^2 a = 1 \Rightarrow A = \frac{1}{\sqrt{a}}$$

$$\text{Also für ungerade Parität: } \Psi_{\text{ungerade}}(x) = \sqrt{\frac{1}{a}} \sin\left(\frac{n\pi}{a} x\right)$$

d) $\psi_1(-a) = \psi_2(a)$ für Fall „gerade“:

$$-A_1 \sin(ka) + B_1 \cos(ka) = 0$$

$$A_2 \sin(ka) + B_2 \cos(ka) = 0$$

$$\Rightarrow B_1 \cos(ka) = A_1 \sin(ka) \Rightarrow \frac{\sin(ka)}{\cos(ka)} = \tan(ka) = \frac{B_1}{A_1}$$

$$A_1 = A_2 = -\frac{mV_0}{\hbar^2 k} B_1 \Rightarrow \boxed{\tan(ka) + \frac{\hbar^2 k}{mV_0} = 0}$$

$$\Rightarrow \tan(ka) + cka = 0 \quad \text{mit } c = \frac{\hbar^2}{mV_0 a}$$

e) $c \ll 1$:

$\Rightarrow -cx$ hat sehr geringe Steigung.

\Rightarrow Schnitt von $\tan(x)$ und $-cx$ ungefähr bei π

$$\tan(x) \approx (x - \pi) + O(x^3)$$

$$\Rightarrow x - \pi + cx = 0 \quad \Rightarrow x = \frac{\pi}{1+c} \quad \Rightarrow k \approx \frac{\pi}{a(1+c)}$$

$$f) \Rightarrow E = \frac{\hbar^2 k^2}{2m} \approx \frac{\hbar^2 \pi^2}{2ma^2 (1+c)^2}$$

$$\frac{1}{(1+c)^2} \approx 1 - 2c \text{ (kleine } c)$$

$$\Rightarrow E \approx \frac{\hbar^2 \pi^2}{2ma^2} (1 - 2c)$$

4.3 Zusatzaufgabe 2

$$a = 1, c = 0.05$$

$$k = \frac{\pi}{1+0.05} \approx 2.991993\dots$$

$$\text{genau: } 2.9930429$$

5 Tutorium vom 06.05.11 Klausur1-Besprechung

5.1 K2

2 Gebiete: $x < 0, x > 0$, jeweils $V(x) = 0$

$$-\frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} = E \psi(x) \Rightarrow \frac{d^2 \psi}{dx^2} = -\frac{2mE}{\hbar^2} \psi$$

$$\kappa := \sqrt{-\frac{2mE}{\hbar^2}}, \quad E < 0$$

Lös.Ansatz (brücksichtigt Normierbarkeit):

$$\psi(x) = \begin{cases} B e^{\kappa x} & , x < 0 \\ A e^{-\kappa x} & x > 0 \end{cases}$$

Bei $x=0$:

i) Wellenfu' stetig $\Rightarrow B = A$

ii) Integration der Schrödingergl. um $x=0$:

$$-\frac{\hbar^2}{2m} \int_{-\varepsilon}^{\varepsilon} \frac{d^2 \psi}{dx^2} dx + \int_{-\varepsilon}^{\varepsilon} V_0 \delta(x) \psi(x) dx = \int_{-\varepsilon}^{\varepsilon} E \psi(x) dx \Rightarrow \frac{\hbar^2}{2m} (\psi'(\varepsilon) - \psi'(-\varepsilon)) = -V_0 \Psi(0) + E(\Psi(\varepsilon) - \Psi(-\varepsilon)) \Psi : \text{Stammfu' von } \psi$$

$\varepsilon \rightarrow 0$: Ableitung von $\psi(x)$ bei $x=0$ springt um $\frac{2mV_0}{\hbar^2} \psi(0)$

$$\text{Es ist } \psi'(x) = \begin{cases} \kappa B e^{\kappa x} & , x < 0 \\ -\kappa A e^{-\kappa x} & x > 0 \end{cases}$$

Somit folgt $\psi'(0^+) - \psi'(0^-) = -\kappa A e^0 - \kappa B e^0$

$$\Rightarrow -\kappa A - \kappa B = \frac{2mV_0}{\hbar^2} A$$

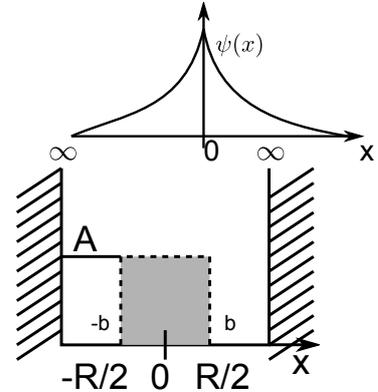
$$\Rightarrow \kappa = -\frac{mV_0}{\hbar^2} = \sqrt{-\frac{2mE}{\hbar^2}} \Rightarrow E = -\frac{mV_0^2}{2\hbar^2}$$

$$\text{Normierung: } \int_{-\infty}^{\infty} |\psi(x)|^2 dx = \int_{-\infty}^0 A^2 e^{2\kappa x} dx + \int_0^{\infty} A^2 e^{-2\kappa x} dx$$

$$= A^2 \underbrace{\left[\frac{e^{\kappa x}}{2\kappa} \right]_{-\infty}^0}_{1/2\kappa} + A^2 \underbrace{\left[-\frac{e^{-2\kappa x}}{2\kappa} \right]_0^{\infty}}_{1/2\kappa} \stackrel{!}{=} 1$$

$$A = \sqrt{\kappa x}$$

$$\psi(x) = \begin{cases} \sqrt{\kappa} e^{\kappa x} & , x < 0 \\ \sqrt{\kappa} e^{-\kappa x} & , x > 0 \end{cases}$$



5.2 K3

$$\psi(x, 0) = A\theta(b^2 - x^2)$$

a) Normierung:

$$\int_{-\frac{R}{2}}^{\frac{R}{2}} dx |\psi(x, 0)|^2 \stackrel{!}{=} 1$$

$$\Rightarrow |A|^2 \cdot 2b = 1 \Rightarrow A = \frac{1}{\sqrt{2b}}$$

b) Spektrum des Potentialtopfes:

$$\phi(\pm \frac{R}{2}) = 0 \Rightarrow \phi(x) = A \cos(kx) + B \sin(kx)$$

$$\text{R.B.: } 0 = \phi(-\frac{R}{2}) = A \cos(k\frac{R}{2}) - B \sin(k\frac{R}{2})$$

$$0 = \phi(\frac{R}{2}) = A \cos(k\frac{R}{2}) + B \sin(k\frac{R}{2})$$

Das Potential ist achsensymmetrisch um $x=0$

$$\text{c) } \det \begin{pmatrix} \cos(z) & -\sin(z) \\ \cos(z) & \sin(z) \end{pmatrix} = 0 \text{ mit } z = \frac{kR}{2}.$$

$$\text{Also } D(z) = 2 \sin(z) \cos(z) \cdot D(z_n) \stackrel{!}{=} 0$$

$$\Rightarrow \sin(z_n) = 0 \text{ bzw. } z_n = n\pi \Rightarrow k_n = \frac{2n\pi}{R}, \quad (n \in \mathbb{N} \geq 1)$$

$$\text{oder } \cos(z_n) = 0 \Rightarrow z_n = (2n+1)\frac{\pi}{2}, \quad k_n = (2n+1)\frac{\pi}{R}$$

$$\Rightarrow E_n = n^2 \frac{\hbar^2}{2m} \left(\frac{\pi}{R}\right)^2 = n^2 E_1 \text{ und } k_n = n \frac{\pi}{R} = nk_1$$

$$\text{Normierung: } \Rightarrow A_n = \sqrt{\frac{2}{R}}$$

d) Also gute Parität:

$$\text{gerade } \phi_n(x) = A_n \cos(k_n x), \quad \phi_n(-x) = +\phi_n(x)$$

$$\text{ungerade: } \phi_n(x) = A_n \sin(k_n x), \quad \phi_n(-x) = -\phi_n(x)$$

e) $\{\varphi_n\}_{n=1, \dots, \infty}$, VONS (vollständiges Orthonormales System)

$$\sum_n |\varphi_n \rangle \langle \varphi_n| = \mathbb{1}$$

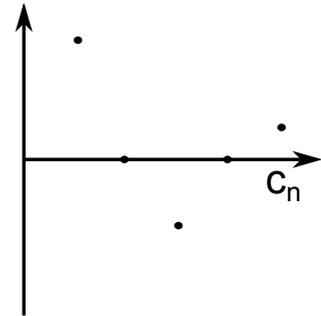
$$\sum_n \varphi_n(x) \varphi_n^*(x') = \delta(x - x')$$

$$\psi(x, 0) = \sum_n \varphi_n(x) \int_{-R/2}^{R/2} dx' \varphi_n^*(x') \psi(x', 0)$$

$$\text{Also gilt: } c_n = \int_{-R/2}^{R/2} dx \varphi_n^*(x) \psi(x, 0)$$

$$c_n = \frac{1}{\sqrt{2b}} \sqrt{\frac{2}{R}} \int_{-b}^b dx \begin{cases} \cos(k_n x) & P = 1 \\ \sin(k_n x) & P = -1 \end{cases}$$

$$= \frac{1}{\sqrt{bR}} \frac{1}{k_n} \begin{cases} \sin(k_n x) \\ \cos(k_n x) \end{cases} \Big|_{-b}^b = \frac{1}{\sqrt{bR}} \frac{1}{k_n} \begin{cases} 2 \sin(k_n b) \\ 0 \end{cases}$$



f) $c_{2n} = 0$

$$c_{2n+1} = \frac{1}{\sqrt{bR}} \frac{2R}{\pi} \frac{1}{2n+1} \sin((2n+1)\pi \frac{b}{R}) \xrightarrow{n \rightarrow \infty} 0$$

$$\text{Wir finden also } \psi(x, 0) = \sum_{n=0}^{\infty} c_{2n+1} \varphi_{2n+1}(x) = \frac{2}{\pi} \sqrt{\frac{R}{b}} \sum_{n \geq 0} \frac{1}{2n+1} \sin((2n+1)\pi \frac{b}{R}) \varphi_{2n+1}(x)$$

g) Für beliebige Zeiten t gilt:

$$\varphi_n(x) \mapsto \phi_n(x, t) = e^{-i\omega_n t} \varphi_n(x)$$

$$\text{und somit } \psi(x, t) = \sum_{n \geq 1} c_{2n+1} \phi_{2n+1}(x)$$

$$\psi(x, t) = \frac{2}{\pi} \sqrt{\frac{R}{b}} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin((2n+1)\pi \frac{b}{R}) e^{-i\omega_{2n+1} t} \varphi_{2n+1}(x)$$

5.3 Zusatz

$$V(x) = \int_0^{\infty} \frac{p \sin(px)}{p^2 + m^2} dp = \frac{1}{2} \int_{-\infty}^{\infty} \frac{p \sin(px)}{p^2 + m^2} dp$$

$$e^{i\phi} = \cos(\phi) + i \sin(\phi) \Rightarrow \sin(px) = \Im(e^{ipx})$$

$$V(x) = \frac{1}{2} \Im \left(\int_{-\infty}^{\infty} \frac{p e^{ipx}}{p^2 + m^2} dp \right)$$

$$\text{Pole bei } p = \pm im \Rightarrow V(x) = \frac{1}{2} \Im(2\pi i \cdot \text{Res}_f(im)) = \frac{1}{2} \Im(2\pi i \cdot \frac{ime^{-mx}}{2im}) = \frac{1}{2} \pi e^{-mx}$$

6 Tutorium vom 03.06.11: Blatt 6

6.1 Hausaufgabe 12

$$\begin{aligned} \text{a) } H &= \frac{p^2}{2m} + \frac{m\omega^2}{2} x^2 - \gamma \sqrt{2m\hbar\omega^3} x \\ &= " + \frac{m\omega^2}{2} (x^2 - 2\gamma \frac{\sqrt{2m\hbar\omega^3} x}{m\omega^2}) \\ &= " + \frac{m\omega^2}{2} (x^2 - 2\gamma \sqrt{\frac{2\hbar}{m\omega}} x) \\ &= " + \frac{m\omega^2}{2} (x^2 - 2\gamma \sqrt{\frac{2\hbar}{m\omega}} x + \gamma^2 \frac{2\hbar}{m\omega} - \gamma^2 \frac{2\hbar}{m\omega}) \\ &= " + \frac{m\omega^2}{2} ((x - \gamma \sqrt{\frac{2\hbar}{m\omega}})^2 - \gamma^2 \frac{2\hbar}{m\omega}) \end{aligned}$$

$$= \frac{p^2}{2m} + \frac{m\omega^2}{2} \left(x - \underbrace{\gamma \sqrt{\frac{2\hbar}{m\omega}}}_{=:x_0} \right)^2 - \gamma^2 \hbar \omega$$

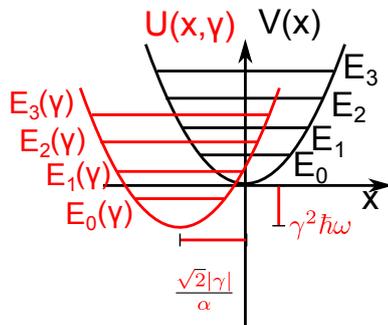
$$x_0 = \gamma \sqrt{\frac{2\hbar}{m\omega}} = \gamma \frac{\sqrt{2}}{\alpha} \quad \text{mit } \alpha = \sqrt{\frac{m\omega}{\hbar}}$$

$$E_n(\gamma) = \hbar\omega \left(n + \frac{1}{2} - \gamma^2 \right)$$

$$\langle x | \psi_n(\gamma) \rangle = \sqrt{\frac{\alpha}{2^n n! \sqrt{\pi}}} \exp\left(-\frac{\alpha(x-x_0)^2}{2}\right) H_n(\alpha(x-x_0))$$

H_n : Hermite Polynome

b)



c) $H_1 = -\gamma \sqrt{2m\hbar\omega^3} x$

$$= -\gamma \sqrt{2m\hbar\omega^3} \sqrt{\frac{\hbar}{2m\omega}} (b^\dagger + b) = -\gamma \hbar \omega (b^\dagger + b)$$

$$\langle n' | H_1 | n \rangle = \langle n' | -\gamma \hbar \omega (b^\dagger + b) | n \rangle = -\gamma \hbar \omega \langle n' | (b^\dagger | n \rangle + b | n \rangle)$$

$$= -\gamma \hbar \omega \langle n' | \sqrt{n+1} | n+1 \rangle - \gamma \hbar \omega \langle n' | \sqrt{n} | n-1 \rangle$$

$$= -\gamma \hbar \omega (\sqrt{n+1} \langle n' | n+1 \rangle + \sqrt{n} \langle n' | n-1 \rangle)$$

$$= -\gamma \hbar \omega (\sqrt{n+1} \delta_{n',n+1} + \sqrt{n} \delta_{n',n-1})$$

7 Tutorium vom 10.06.11 (Blatt 7)

7.1 Hausaufgabe 13

a) $[\sigma_i, \sigma_j] = 2i \varepsilon_{ijk} \sigma_k$

Alle aufführen:

$$[\sigma_1, \sigma_2] = \sigma_1 \sigma_2 - \sigma_2 \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} i & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} = 2i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2i \sigma_3$$

$$[\sigma_2, \sigma_3] = 2i \sigma_1 \varepsilon_{231} = 2i \sigma_1$$

$$[\sigma_1, \sigma_3] = 2i \sigma_2 \varepsilon_{132} = -2i \sigma_2$$

etc.

$$\{\sigma_i, \sigma_j\} = \sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij} \mathbb{1} = 2\delta_{ij} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\{\sigma_1, \sigma_2\} = \sigma_1\sigma_2 + \sigma_2\sigma_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\{\sigma_1, \sigma_1\} = 2\sigma_1\sigma_1 = 2\mathbb{1}$$

b) $[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k = \sigma_i\sigma_j - \sigma_j\sigma_i$

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{1} = \sigma_i\sigma_j + \sigma_j\sigma_i$$

$$\text{,,+“} \Rightarrow 2\sigma_i\sigma_j = 2(\delta_{ij}\mathbb{1} + \varepsilon_{ijk}\sigma_k)$$

$$\sigma_i\sigma_j = \delta_{ij}\mathbb{1} + i\varepsilon_{ijk}\sigma_k$$

$$(\vec{a}\vec{\sigma})(\vec{b}\vec{\sigma}) = \vec{a}\vec{b}\mathbb{1} + i\sigma(\vec{a} \times \vec{b})$$

$$= a_i\sigma_i b_j\sigma_j = a_i b_j \sigma_i\sigma_j = a_i b_j (\delta_{ij}\mathbb{1} + i\varepsilon_{ijk}\sigma_k) = a_i b_j \delta_{ij} + i\varepsilon_{ijk} a_i b_j \sigma_k = (\vec{a}\vec{b})\mathbb{1} + i\vec{\sigma}(\vec{a} \times \vec{b})$$

c) $e^{i\alpha\sigma_k} = \cos(\alpha) + i\sigma_k \sin(\alpha), \quad \alpha \in \mathbb{C}, \quad k = 1, 2, 3$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\begin{aligned} e^{i\alpha\sigma_k} &= \sum_{n=0}^{\infty} \frac{i^n \alpha^n}{n!} \sigma_k^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n} \sigma_k^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n (\alpha\sigma_k)^{2n+1}}{(2n+1)!} \\ &= \mathbb{1} \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n}}{(2n)!} + i\sigma_k \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n+1}}{(2n+1)!} \\ &= \mathbb{1} \cos(\alpha) + i\sigma_k \sin(\alpha) \end{aligned}$$

d) $M = m_0\mathbb{1} + \sum_{i=1,2,3} m_i\sigma_i = \begin{pmatrix} m_0 & 0 \\ 0 & m_0 \end{pmatrix} + \begin{pmatrix} 0 & m_1 \\ m_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -im_2 \\ im_2 & 0 \end{pmatrix} + \begin{pmatrix} m_3 & 0 \\ 0 & -m_3 \end{pmatrix} = \begin{pmatrix} m_0 + m_3 & m_1 - im_2 \\ m_1 + im_2 & m_0 - m_3 \end{pmatrix}$

$$m_0 = \frac{\text{tr}(M)}{2}, \quad m_i = \frac{1}{2}\text{tr}(\sigma_i M)$$

e) trivial

f) i) $\sigma_i^\dagger = (\sigma_i^*)^T$ hermitisch

$$\sigma_2^\dagger = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}^T = \sigma_1$$

$$\sigma_3^\dagger = \sigma_1$$

ii) $\text{tr}(\sigma_i) = 0$

iii) $\det(\sigma_i) = -1$

iv) $\sigma_i^T = (-1)^{i+1}\sigma_i$

7.2 Hausaufgabe 14

$$\phi(x) = A x e^{-\frac{x^2}{2\gamma^2}}$$

$$\begin{aligned} \text{a) } \int_{-\infty}^{\infty} dx |\phi(x)|^2 = 1 &\Rightarrow \frac{1}{|A|^2} = \int_{-\infty}^{\infty} dx x^2 e^{-\frac{x^2}{\gamma^2}} = -\frac{\partial}{\partial \alpha} \Big|_{\frac{1}{\gamma^2}} \int_{-\infty}^{\infty} dx e^{-\alpha x^2} = -\frac{\partial}{\partial \alpha} \Big|_{\frac{1}{\gamma^2}} \sqrt{\frac{\pi}{\alpha}} = \\ & \left[\frac{1}{2} \frac{\sqrt{\pi}}{\alpha^{3/2}} \right]_{\alpha=\frac{1}{\gamma^2}} \\ & = \gamma^3 \cdot \frac{1}{2} \sqrt{\pi} \Rightarrow A = \sqrt{\frac{2}{\gamma^3 \sqrt{\pi}}} \end{aligned}$$

$$\text{b) } (H - E)\phi(x) = 0 = (T + V(x) - E)\phi(x)$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{2m}{\hbar^2} (E - V(x)) \right) \phi(x) = 0$$

$$\phi' = A \left(1 - \frac{x^2}{\gamma^2} \right) e^{-\frac{x^2}{2\gamma^2}}$$

$$\phi'' = A \left[-\frac{2x}{\gamma^2} - \frac{x}{\gamma^2} \left(1 - \frac{x^2}{\gamma^2} \right) \right] e^{-\frac{x^2}{2\gamma^2}}$$

$$= A \left[-\frac{3}{\gamma^2} x + \frac{1}{\gamma^4} x^3 \right] e^{-\frac{x^2}{2\gamma^2}}$$

$$= - \left[\frac{3}{\gamma^2} - \frac{x^2}{\gamma^4} \right] A x e^{-\frac{x^2}{2\gamma^2}}$$

$$\phi'' = - \left[\frac{3}{\gamma^2} - \frac{x^2}{\gamma^4} \right] \phi(x)$$

$$\text{d.h. } \left[\frac{\partial^2}{\partial x^2} + \left(\frac{3}{\gamma^2} - \frac{x^2}{\gamma^4} \right) \right] \phi(x) = 0$$

$$\text{und damit } \frac{2m}{\hbar^2} (E - V(x)) = \frac{3}{\gamma^2} - \frac{x^2}{\gamma^4}$$

$$\text{Mit } V(0) = 0: \quad E = \frac{3}{2} \frac{\hbar^2}{m\gamma^2}, \quad V(x) = \frac{\hbar^2}{2m} \frac{x^2}{\gamma^4}$$

$$\text{Außerdem erkennt man: } \frac{1}{\gamma^2} = \frac{m\omega}{\hbar} \Rightarrow E = \frac{3}{2} \hbar\omega = [\hbar\omega(n + \frac{1}{2})]_{n=1}$$

$$\Rightarrow V(x) = \frac{1}{2} \hbar\omega \frac{x^2}{\gamma^2}$$

$\Rightarrow \phi(x)$ ist der 1. angeregte Zustand in einem Oszillatorpotential

8 Tutorium vom 17.06.11 (Blatt 8)

PM

$$L_x = \begin{pmatrix} 0 & \hbar/\sqrt{2} & 0 \\ \hbar/\sqrt{2} & 0 & \hbar/\sqrt{2} \\ 0 & \hbar/\sqrt{2} & 0 \end{pmatrix}$$

a)

$$P_{L_x}(\lambda) = \det(L_x - \lambda \cdot \mathbb{1}_3) \quad \text{charakt. Polynom}$$

$$= \det \begin{pmatrix} -\lambda & \hbar/\sqrt{2} & 0 \\ \hbar/\sqrt{2} & -\lambda & \hbar/\sqrt{2} \\ 0 & \hbar/\sqrt{2} & -\lambda \end{pmatrix}$$

Entwicklung nach 1. Zeile

$$= -\lambda \begin{vmatrix} -\lambda & \hbar/\sqrt{2} \\ \hbar/\sqrt{2} & -\lambda \end{vmatrix} - \frac{\hbar}{\sqrt{2}} \begin{vmatrix} \hbar/\sqrt{2} & \hbar/\sqrt{2} \\ 0 & -\lambda \end{vmatrix}$$

$$= -\lambda \left(\lambda^2 - \frac{\hbar^2}{2} \right) - \frac{\hbar}{\sqrt{2}} \cdot \left(-\lambda \cdot \frac{\hbar}{\sqrt{2}} \right)$$

$$= -\lambda^3 + \lambda \frac{\hbar^2}{2} + \lambda \frac{\hbar^2}{2}$$

$$= -\lambda^3 + \lambda \hbar^2 = \lambda(-\lambda^2 + \hbar^2) \stackrel{!}{=} 0$$

$$\Rightarrow \lambda_1 = 0$$

$$\Rightarrow \lambda_2 = +\hbar$$

$$\Rightarrow \lambda_3 = -\hbar$$

} Eigenwerte

Bestimmung der Eigenvektoren:

b)

$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ ist Eigenvektor von L_{x_1} wenn \vec{x} nichttriviale

Lösung von $(L_x - \lambda \mathbb{1}_3) \vec{x} = \vec{0}$ ist, d.h. für $\lambda = 0$:

$$\begin{matrix} \rightarrow \\ \rightarrow \end{matrix} \begin{pmatrix} 0 & \hbar/\sqrt{2} & 0 \\ \hbar/\sqrt{2} & 0 & \hbar/\sqrt{2} \\ 0 & \hbar/\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{Zeilen tauschen}$$

$$+ \begin{matrix} \cdot (-1) \\ \rightarrow \end{matrix} \begin{pmatrix} \hbar/\sqrt{2} & 0 & \hbar/\sqrt{2} & : & 0 \\ 0 & \hbar/\sqrt{2} & 0 & : & 0 \\ 0 & \hbar/\sqrt{2} & 0 & : & 0 \end{pmatrix} \quad \text{Gauß-Algorithmus zum Lösen des LGS}$$

$$\begin{pmatrix} \hbar/\sqrt{2} & 0 & \hbar/\sqrt{2} & : & 0 \\ 0 & \hbar/\sqrt{2} & 0 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{pmatrix} \Rightarrow \frac{\hbar}{\sqrt{2}} x_1 = -\frac{\hbar}{\sqrt{2}} x_3 \Rightarrow \boxed{-x_1 = x_3}$$

$$\Rightarrow \frac{\hbar}{\sqrt{2}} \cdot x_2 = 0 \Rightarrow \boxed{x_2 = 0}$$

Sei $x_3 = t$. Dann ist der Eigenvektor also von der Form $(-t, 0, t)$. Der normierte Eigenvektor lautet also $(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$ und gehört zum Eigenwert $\lambda_1 = 0$.

Analog erhält man durch das Lösen von LGS:

Eigenwert

normierter Eigenvektor

$-\hbar$

$$\left(\frac{1}{2}, -\frac{1}{\sqrt{2}}, \frac{1}{2}\right) =: \vec{a}_1$$

0

$$\left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) =: \vec{a}_2$$

$+\hbar$

$$\left(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right) =: \vec{a}_3$$

c)

Drei Eigenvektoren bilden ein Orthonormalsystem:

$$\vec{a}_1 \cdot \vec{a}_2 = \frac{1}{2} \cdot \left(-\frac{1}{\sqrt{2}}\right) + 0 + \frac{1}{2} \cdot \frac{1}{\sqrt{2}} = 0$$

$$\vec{a} \cdot \vec{b} = 0 \Rightarrow \vec{a} \perp \vec{b}$$

$$\vec{a}_1 \cdot \vec{a}_3 = \frac{1}{2} \cdot \frac{1}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} - \frac{1}{2} + \frac{1}{4} = 0$$

$$\vec{a}_2 \cdot \vec{a}_3 = -\frac{1}{\sqrt{2}} \cdot \frac{1}{2} + 0 + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = 0$$

d) Bezüglich der Basis $B = \{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ haben wir

$$L_x^B = \begin{pmatrix} \vec{a}_1 \cdot L_x \cdot \vec{a}_1 & \vec{a}_1 \cdot L_x \cdot \vec{a}_2 & \vec{a}_1 \cdot L_x \cdot \vec{a}_3 \\ \vec{a}_2 \cdot L_x \cdot \vec{a}_1 & \vec{a}_2 \cdot L_x \cdot \vec{a}_2 & \vec{a}_2 \cdot L_x \cdot \vec{a}_3 \\ \vec{a}_3 \cdot L_x \cdot \vec{a}_1 & \vec{a}_3 \cdot L_x \cdot \vec{a}_2 & \vec{a}_3 \cdot L_x \cdot \vec{a}_3 \end{pmatrix}$$

$$L_x \cdot \vec{a}_1 = \begin{pmatrix} 0 & \hbar/\sqrt{2} & 0 \\ \hbar/\sqrt{2} & 0 & \hbar/\sqrt{2} \\ 0 & \hbar/\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{pmatrix} = \begin{pmatrix} -\hbar/2 \\ \hbar/\sqrt{2} \\ -\hbar/2 \end{pmatrix}$$

$$\vec{a}_1(L_x \cdot \vec{a}_1) = \left(\frac{1}{2}, -\frac{1}{\sqrt{2}}, \frac{1}{2}\right) \cdot \left(-\frac{\hbar}{2}, \frac{\hbar}{\sqrt{2}}, -\frac{\hbar}{2}\right)$$

$$= -\frac{\hbar}{4} - \frac{\hbar}{2} - \frac{\hbar}{4} = -\hbar = (L_x^B)_{11}$$

Durch 8 analoge Rechnungen findet man schließlich

$$L_x^B = \begin{pmatrix} -\hbar & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \hbar \end{pmatrix}$$

H15

$$x, y, z \in \mathbb{R}^+,$$

$$x + y + z \leq 3.$$

$$\text{Sei } x_1 = \sqrt{x}, \quad y_1 = \sqrt{\frac{1}{x}},$$

$$x_2 = \sqrt{y}, \quad y_2 = \sqrt{\frac{1}{y}},$$

$$x_3 = \sqrt{z}, \quad y_3 = \sqrt{\frac{1}{z}}.$$

$$\text{Cauchy-Schwarz: } (x_1^2 + x_2^2 + x_3^2)(y_1^2 + y_2^2 + y_3^2) \geq (x_1 y_1 + x_2 y_2 + x_3 y_3)^2$$

für reelle x_i und y_j

$$\text{Es ist } (x_1 y_1 + x_2 y_2 + x_3 y_3)^2 = (1 + 1 + 1)^2 = 3^2 = 9.$$

$$\text{Also folgt } 9 \leq (x + y + z) \cdot \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right)$$

Teile diese Ungleichung durch $x + y + z$ (positive Zahl $\neq 0$).

$\frac{9}{x + y + z}$ ist größer oder gleich 3, weil der Nenner höchstens 3 ist.

$$\text{Also: } 3 \leq \frac{9}{x + y + z} \leq \frac{1}{x} + \frac{1}{y} + \frac{1}{z},$$

was zu beweisen war.

H16) a), b), c): viele Matrixrechnungen, siehe extra pdf mit Mathematica-Rechnungen

d), e), f): analog Präsenzaufgabe, siehe extra pdf.

Zusatzaufgabe

$$\text{Es ist } a = \sqrt{\frac{m\omega}{2\hbar}} x + i \cdot \frac{1}{\sqrt{2\hbar m\omega}} \cdot \frac{\hbar}{i} \frac{\partial}{\partial x}$$

$$a = \sqrt{\frac{m\omega}{2\hbar}} Q + i \frac{P}{\sqrt{2\hbar m\omega}} ; \quad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}} Q - i \frac{P}{\sqrt{2\hbar m\omega}}$$

Paritätoperator \hat{P} :

$$\begin{aligned} \hat{P}a &= \sqrt{\frac{m\omega}{2\hbar}} (-x) + i \cdot \frac{1}{\sqrt{2\hbar m\omega}} \frac{\hbar}{i} \left(-\frac{\partial}{\partial x}\right) \\ &= - \left(\sqrt{\frac{m\omega}{2\hbar}} x + i \frac{1}{\sqrt{2\hbar m\omega}} \frac{\hbar}{i} \frac{\partial}{\partial x} \right) = -a \end{aligned}$$

\Rightarrow negative Parität

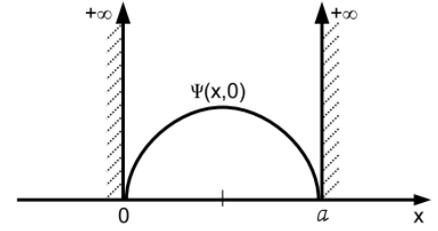
$$\begin{aligned} \hat{P}a^\dagger &= \sqrt{\frac{m\omega}{2\hbar}} (-x) - i \frac{1}{\sqrt{2\hbar m\omega}} \frac{\hbar}{i} \left(-\frac{\partial}{\partial x}\right) \\ &= - \sqrt{\frac{m\omega}{2\hbar}} x + i \frac{1}{\sqrt{2\hbar m\omega}} \frac{\hbar}{i} \frac{\partial}{\partial x} \\ &= - \left(\sqrt{\frac{m\omega}{2\hbar}} x - i \frac{1}{\sqrt{2\hbar m\omega}} \frac{\hbar}{i} \frac{\partial}{\partial x} \right) = -a^\dagger \Rightarrow \text{negative Parität} \end{aligned}$$

Aufgrund der negativen Parität kann es keine von 0 verschiedenen Diagonalmatrixelemente geben.

9 Tutorium vom 24.06.11 (Blatt 9)

9.1 Hausaufgabe 17

$$\psi(x, 0) = Ax(a - x)$$



$$\begin{aligned} \text{a) } 1 &= A^2 \int_0^a dx x^2 (a - x)^2 = A^2 \int_0^a dx x^2 (a^2 - 2ax + x^2) \\ &= A^2 \int_0^a dx (a^2 x^2 - 2ax^3 + x^4) = A^2 \left(\frac{a^5}{3} - \frac{2}{4} a^5 + \frac{1}{5} a^5 \right) \\ &= A^2 a^5 \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right) = A^2 a^5 \frac{10 - 15 + 6}{30} = A^2 \frac{a^5}{30} \\ &\Rightarrow A = \sqrt{\frac{30}{a^5}} = \frac{1}{a^2} \sqrt{\frac{30}{a}} \\ \psi(x, 0) &= (\pm) \sqrt{\frac{30}{a^5}} x(a - x) \end{aligned}$$

b) Entweder nach VON des Topfes:

$$\phi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right), \quad n = 1, 2, \dots, \quad 0 \leq x \leq a$$

$$\psi(x, 0) = \sum_n c_n \phi_n(x)$$

$$c_n = \sqrt{\frac{2}{a}} A \int_0^a dx x(a - x) \sin(k_n x), \quad k_n = \frac{n\pi}{a}$$

$$= A \sqrt{\frac{2}{a}} \int_0^a dx (ax - x^2) \sin(k_n x)$$

$$\begin{aligned} 1) \int_0^a dx x \sin(k_n x) &= -\frac{1}{k_n} \int_0^a dx x \frac{d}{dx} \cos(k_n x) \\ &= -\frac{1}{k_n} a \cos(k_n a) + \frac{1}{k_n} \int_0^a dx \cos(k_n x) = -\frac{1}{k_n} a \cos(k_n a) + \frac{1}{k_n^2} \underbrace{\sin(k_n a)}_{=0} \\ &= -\frac{1}{k_n} a \cos(n\pi) = -(-1)^n \frac{a^2}{n\pi} \end{aligned}$$

$$\begin{aligned} 2) \int_0^a dx x^2 \sin(k_n x) &= -\frac{\partial^2}{\partial k_n^2} \int_0^a dx \sin(k_n x) \\ \stackrel{s=k_n a=n\pi}{=} \frac{\partial^2}{\partial k_n^2} \frac{1}{k_n} (\cos(k_n a) - 1) &= a^3 \frac{d^2}{ds^2} \left(\frac{\cos(s) - 1}{s} \right) \\ \frac{d^2}{dx^2} (g(x)h(x)) &= g''h + 2h'g' + hg'' \\ \Rightarrow \frac{d^2}{ds^2} \frac{\cos(s) - 1}{s} &= \frac{2}{s^3} (\cos(s) - 1) + \frac{2}{s^2} \sin(s) - \frac{1}{s} \\ \frac{2}{(n\pi)^3} ((-1)^n - 1) - \frac{1}{n\pi} (-1)^n &= \int_0^a dx x^2 \sin(k_n x) \frac{1}{a^3} \end{aligned}$$

$$= \begin{cases} -\frac{1}{n\pi} & , n = 2k, k \in \mathbb{N} \\ -\frac{4}{(n\pi)^3} + \frac{1}{n\pi} & , n = 2k + 1, k \in \mathbb{N} \end{cases}$$

$$\text{Damit: } c_n = \sqrt{\frac{2}{a}} A a^3 \left[(-1)^{n+1} \left(\frac{1}{n\pi} - \frac{1}{n\pi} \right) + \frac{4}{(n\pi)^3} \frac{1}{2} (1 - (-1)^n) \right]$$

$$= \sqrt{\frac{2}{a}} a^3 A \frac{4}{n^3 \pi^3} \frac{1}{2} (1 - (-1)^n)$$

$$c_{2k+1} = \sqrt{60} \frac{4}{\pi^3} \frac{1}{(2k+1)^3}, \quad c_{2k} = 0$$

$$\psi(x, 0) = \sqrt{15} \frac{8}{\pi^3} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} \phi_{2k+1}(x)$$

c) Berechnung der Wahrscheinlichkeiten:

$$\omega_1 = |c_1|^2 = \frac{15 \cdot 64}{\pi^6} = \frac{16 \cdot 60}{\pi^6} = \frac{960}{\pi^6} \approx 0.9986$$

$$\omega_1 = 1 - \bar{\omega} = 1.44 \cdot 10^{-3}$$

Norm der Reihe:

$$\int_0^a dx |\psi(x, 0)|^2 = \frac{960}{\pi^6} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^6} = \frac{(2^6-1)\pi^6}{6! \cdot 2} \frac{960}{\pi^6} B_6 = \frac{63\pi^6}{720 \cdot 2} \frac{1}{42} \frac{960}{\pi^6} = \frac{63 \cdot 60}{45 \cdot 2 \cdot 42} = 1$$

d) Zeitentwicklung:

$$\psi(x, t) = \frac{\sqrt{960}}{\pi^3} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} \phi_{2k+1} e^{-i\omega_{2k+1}t}$$

$$\omega_n = \frac{1}{\hbar} E_n = \frac{\hbar}{2m} \frac{\pi^2}{a^2} n^2$$

e) Wahrscheinlichkeitsdichte, -stromdichte:

$$\rho(x, t) = \sum_{n,k} c_{2k+1}^* \varphi_{2k+1}^*(x) c_{2n+1} \varphi_{2n+1}(x) e^{i(\omega_{2k+1} - \omega_{2n+1})t}$$

$$c_j, \varphi_j(x) \in \mathbb{R}$$

$$\rho(x, t) = \rho_s(x) + \rho_t(x, t)$$

$$\rho_s(x) = \sum_n c_{2n+1}^2 \varphi_{2n+1}^2(x)$$

$$\rho_t(x, t) = \sum_{k < n} c_{2k+1} c_{2n+1} \varphi_{2k+1}(x) \varphi_{2n+1}(x) (e^{i(\omega_{2k+1} - \omega_{2n+1})t} + e^{-i(\omega_{2k+1} - \omega_{2n+1})t})$$

$$= 2 \sum_{k < n} c_{2k+1} c_{2n+1} \varphi_{2k+1}(x) \varphi_{2n+1}(x) \cos((\omega_{2k+1} - \omega_{2n+1})t)$$

$$j(x, t) = \frac{\hbar}{2im} \sum_{n,k} (\varphi_{2k+1}^*(x) \frac{d}{dx} \varphi_{2n+1}(x) - \varphi_{2n+1} \frac{d}{dx} \varphi_{2k+1}^*(x)) c_{2k+1} c_{2n+1} e^{i(\omega_{2k+1} - \omega_{2n+1})t}$$

$$= \frac{\hbar}{2im} \sum_{k < n} (\varphi_{2n+1}(x) \frac{d}{dx} \varphi_{2n+1}(x) - \varphi_{2n+1}(x) \frac{d}{dx} \varphi_{2k+1}(x)) c_{2k+1} c_{2n+1} (e^{i(\omega_{2k+1} - \omega_{2n+1})t} - e^{-i(\omega_{2k+1} - \omega_{2n+1})t})$$

$$= \frac{\hbar}{m} \sum_{k < n} (\varphi_{2n+1}(x) \frac{d}{dx} \varphi_{2n+1}(x) - \varphi_{2n+1}(x) \frac{d}{dx} \varphi_{2k+1}(x)) c_{2k+1} c_{2n+1} \sin((\omega_{2k+1} - \omega_{2n+1})t)$$

$$\frac{\partial}{\partial t} \rho(x, t) + \frac{\partial}{\partial x} j(x, t) = 0 \text{ Kontinuitätsgl. ist erfüllt:}$$

$$\frac{\partial}{\partial t} \rho(x, t) = \frac{\partial}{\partial t} \rho_t(x, t)$$

$$= -\frac{2}{\hbar} \sum_{k < n} (E_{2k+1} - E_{2n+1}) c_{2k+1} c_{2n+1} \varphi_{2k+1}(x) \varphi_{2n+1}(x) \sin((\omega_{2k+1} - \omega_{2n+1})t)$$

und mit $\varphi_j''(x) = -q_j^2 \varphi_j(x)$ ergibt sich:

$$\frac{\partial}{\partial x} j(x, t) = \frac{\hbar}{m} \sum_{k < n} (q_{2k+1}^2 - q_{2n+1}^2) \varphi_{2k+1}(x) \varphi_{2n+1}(x) c_{2k+1} c_{2n+1} \sin((\omega_{2k+1} - \omega_{2n+1})t)$$

Also:

$$\frac{\partial}{\partial x} j(x, t) = \frac{2}{\hbar} \sum_{k < n} (E_{2k+1} - E_{2n+1}) \varphi_{2k+1}(x) \varphi_{2n+1}(x) c_{2k+1} c_{2n+1} \sin((\omega_{2k+1} - \omega_{2n+1})t)$$

f) Berechnung von $\langle x \rangle (t)$:

$$\langle x \rangle (t) = \int_0^a dx \psi^*(x, t) x \psi(x, t)$$

$$x = (x - \frac{a}{2}) + \frac{a}{2}$$

$$\langle x \rangle (t) = \frac{960}{\pi^6} \sum_{k, k'} \frac{1}{(2k+1)^3 (2k'+1)^3} \left[\frac{a}{2} \delta_{kk'} + \int_0^a dx (x - \frac{a}{2}) \phi_{2k+1}(x) \phi_{2k'+1}(x) e^{i(\omega_{2k'+1} - \omega_{2k+1})t} \right]$$

Da nur Funktionen auftreten, die Symmetrisch um $x = \frac{a}{2}$ sind, verschwindet

das 2. Integral und $\langle x \rangle (t)$ ist stationär.

$$\langle x \rangle (t) = \frac{960}{\pi^6} \frac{a}{2} \sum_k \frac{1}{(2k+1)^6} = \frac{a}{2}$$

Schwankung der Energie:

$$\begin{aligned} (\Delta E)^2 &= \langle (H - \langle H \rangle)^2 \rangle = \langle H^2 - \langle H \rangle^2 \rangle \\ \langle H \rangle &= \int_0^a dx \psi^*(x, t) H \psi(x, t) = -i\hbar \int_0^a dx \psi^*(x, t) \frac{\partial}{\partial t} \psi(x, t) \\ &= \frac{960}{\pi^6} \sum_k \frac{1}{(2k+1)^6} E_{2k+1} = \frac{960}{\pi^6} \frac{\hbar^2}{2m} \frac{\pi^2}{a^2} \sum_k \frac{1}{(2k+1)^4} = E_1 \frac{960}{\pi^6} \frac{(2^4-1)\pi}{4!2} |B_4| \\ &= E_1 \frac{960}{\pi^2} \frac{15}{48 \cdot 30} = \frac{10}{\pi^2} E_1 \approx 1.01 E_1 \end{aligned}$$

$$\text{Zu } \langle H^2 \rangle: H^2 \phi_n = H(H\phi_n) = E_n H\phi_n = E_n^2 \phi_n$$

$$\begin{aligned} \langle H^2 \rangle &= \frac{960}{\pi^6} E_1^2 \sum_{k=0}^{\infty} \frac{(2k+1)^4}{(2k+1)^6} = \frac{960}{\pi^6} E_1^2 \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \\ &= \frac{960}{\pi^6} E_1^2 \frac{(2^2-1)\pi^2}{2! \cdot 2} |B_2| E_1^2 \frac{960}{\pi^4} \frac{3}{4 \cdot 6} \approx 1.23 E_1^2 \end{aligned}$$

$$\Delta E = \sqrt{\langle H^2 \rangle - \langle H \rangle^2} = E_1 \sqrt{\frac{120}{\pi^4} - \frac{100}{\pi^4}} = E_1 \frac{\sqrt{20}}{\pi^2}$$

$$\Delta E = 2E_1 \frac{\sqrt{5}}{\pi^2} \approx 0.453 E_1$$

10 Tutorium vom 01.07.11

10.1 Z5

$$\text{a) } G = \frac{1}{E+i\eta-H} \quad \eta \rightarrow 0+$$

$$H = \frac{\vec{p}^2}{2m} = -\frac{\hbar^2}{2m} \vec{\nabla}^2 = -\frac{\hbar^2}{2m} \frac{d^2}{dr^2}$$

$$H\psi = e\psi$$

$$-\hbar^2/2m \frac{d^2}{dr^2} \psi = e\psi$$

$$\psi'' + \frac{2me}{\hbar^2} \psi = 0$$

$$k^2 = \frac{2me}{\hbar^2} \Rightarrow e = \frac{\hbar^2 k^2}{2m} = e(k)$$

$$\psi(\vec{r}) = e^{i\vec{k}\vec{r}}$$

$$\langle \vec{k} | \vec{k}' \rangle = \int \delta^3(\vec{k} - \vec{k}') d^3k$$

$$\langle \vec{k} | \vec{k} \rangle = 1$$

$$\langle \vec{r} | \vec{k} \rangle = \frac{1}{(2\pi)^{3/2}} e^{i\vec{k}\vec{r}}$$

$$= \langle \vec{r} | \frac{1}{E+i\eta-H} | \vec{r}' \rangle = \int d\vec{k} \langle \vec{r} | \vec{k} \rangle \frac{1}{E+i\eta-e} \langle \vec{k} | \vec{r}' \rangle = \int \frac{1}{(2\pi)^3} d^3k e^{i\vec{k}\vec{r}} \frac{1}{E+i\eta-e(k)} e^{-i\vec{k}\vec{r}'} =$$

$$\int \frac{1}{(2\pi)^3} d^3k e^{i\vec{k}(\vec{r}-\vec{r}')} \frac{1}{E+i\eta-e(k)}$$

$$\text{b) } \int \frac{1}{(2\pi)^3} d^3k e^{i\vec{k}(\vec{r}-\vec{r}')} \frac{1}{E+i\eta-e(k)} r - \vec{r}' = \vec{s}, \quad \vec{k} = k\vec{e}_z$$

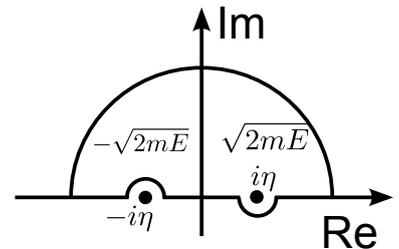
$$\text{sphärische Koord: } \dots = \frac{1}{(2\pi)^3} \int_0^\infty \int_0^{2\pi} \int_{-1}^1 dk k^2 d\cos(\theta) d\varphi e^{iks \cos(\theta)} \frac{1}{E+i\eta-e(k)}$$

$$= \frac{1}{4\pi^3} \int_{-\infty}^0 dk k^2 \frac{e^{iks} - e^{-iks}}{iks} \frac{1}{E+i\eta-e(k)}$$

$$\begin{aligned}
&= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dk \frac{k e^{iks}}{is} \frac{1}{E+i\eta-e(k)} = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dk \frac{k e^{iks}}{is} \frac{1}{E+i\eta-\frac{\hbar^2 k^2}{2m}} \\
&= \frac{m}{2\pi^2 \hbar^2} \int_{-\infty}^{\infty} dk \frac{k e^{iks}}{is} \frac{1}{\frac{2m}{\hbar^2} E+i\eta-k^2} \\
&= \left. \begin{aligned} E > 0, \text{ dann } \frac{2mE}{\hbar^2} - k^2 &= 0 \\ \hbar k_{1,2} &= \pm \sqrt{2mE + i\eta} \\ (\eta \rightarrow 0_+) \quad \hbar k_1 &= \sqrt{2mE} + i\eta \\ \hbar k_2 &= -\sqrt{2mE} + i\eta \end{aligned} \right\}
\end{aligned}$$

c) $\hbar k_1 = \sqrt{2mE} + i\eta$ lies inside the contour

$$\begin{aligned}
&\Rightarrow \frac{m}{2\hbar^2 \pi^2} \int_{-\infty}^{\infty} dk \frac{k}{\frac{2mE}{\hbar^2} - k^2} \frac{e^{iks}}{is} \\
&= \frac{m}{2\hbar^2 \pi^2} \cdot 2\pi i \operatorname{Res}_{\sqrt{2mE}} \left(\frac{k}{\frac{2mE}{\hbar^2} - k^2} - \frac{e^{iks}}{is} \right) \\
&= -\frac{m}{2\pi \hbar} \frac{e^{is \frac{\sqrt{2mE}}{\hbar}}}{s} \\
G(\vec{r}, \vec{r}') &= -\frac{m}{2\pi \hbar^2} \exp\left(\frac{i\sqrt{2mE}}{\hbar}(\vec{r} - \vec{r}')\right) \\
G(s) &\sim -\frac{e^{is}}{s} \sim \frac{\cos(s)}{s} + i \frac{\sin(s)}{s}
\end{aligned}$$



which exhibits the outgoing wave behavior from the sourcepoint: $\vec{r}' \rightarrow \vec{r}$

10.2 Minitest 6

a) $\{|a_1\rangle, |a_2\rangle\}$, $\langle a_i | a_j \rangle = \delta_{ij}$

$$P_i = |a_i\rangle \langle a_i|$$

$$P_i^2 |a_i\rangle = \underbrace{\langle a_i | a_i \rangle}_{=1} \langle a_i | a_i \rangle |a_i\rangle = |a_i\rangle \langle a_i| |a_i\rangle = P_i |a_i\rangle$$

b) $p_i^\dagger = (|a_i\rangle \langle a_i|)^\dagger = P_i$

c) def.: $|a_i\rangle \rightarrow \vec{e}_i = \begin{pmatrix} \langle a_1 | e \rangle \\ \langle a_2 | e \rangle \end{pmatrix}$

Im R^2 : (2×2) - Matrix

$$\langle a_i | \rightarrow (|a_i\rangle)^\dagger \rightarrow | \langle a_i | e \rangle^* \langle a_2 | e \rangle^* \rangle$$

$$a_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$P_1 = |a_1\rangle \langle a_1| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

10.3 Hausaufgabe 18

a) $x\phi_s(x) = x\delta(x-s) = s\phi$, also EW: $x = s$

b) Die stationäre Impulseigenfu' sind bekannt: ebene Wellen

$\phi_k(x) = e^{ikx}$. Und es gilt bekanntlich

$$\phi_s(x) = \delta(x-s) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-s)} = \int_{-\infty}^{\infty} \frac{dk}{2\pi} c_k(s) e^{iks}$$

Also EW-Koeff $c_k(s) = e^{-iks}$

Kerim's Lösung:

$$\begin{aligned} \text{a) } x\delta(x-s) &= \lambda\delta(x-s) \quad \Big| \int_{-\infty}^{\infty} dx \\ \underbrace{\int_{-\infty}^{\infty} x\delta(x-s)dx}_s &= \underbrace{\int_{-\infty}^{\infty} \lambda\delta(x-s)dx}_{=\lambda} \\ \Rightarrow EW : \lambda &= s \end{aligned}$$

b) $\hat{\partial}_x \phi(x) = p\phi(x)$

$$-i\hbar \frac{\partial}{\partial x} \phi(x) - p\phi(x) = 0 \Leftrightarrow \phi'(x) + \frac{p}{i\hbar} \phi(x) = 0$$

$$\Rightarrow \phi(x) = \phi_p(x) = e^{i\frac{p}{\hbar}x}, \quad p \in \mathbb{R}$$

$$\phi_s(x) = \delta(x-s) = \int_{-\infty}^{\infty} \hat{\phi}_s(k) e^{iks} dk$$

$$\text{FT: } \hat{\phi}_s(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_s(x) e^{-ikx} dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(x-s) e^{-ikx} dx = \frac{1}{2\pi} e^{-iks}$$

$$\phi_s(x) = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-iks} e^{ikx} dk \underset{k=\frac{p}{\hbar}}{=} \int_{-\infty}^{\infty} \frac{1}{2\pi\hbar} e^{-i\frac{p}{\hbar}s} \underbrace{e^{i\frac{p}{\hbar}x}}_{\phi_p(x)} dp$$

Der Entwicklungskoeffizient ist somit $\frac{1}{2\pi\hbar} e^{-i\frac{p}{\hbar}s}$

11 Tutorium vom 08.07.2011 (2. Klausur)

11.1 K2

$$\text{a) } V(r) = -\frac{Ze^2}{r}$$

$$\text{Schrödinger: } (T + V(r) - E_0)\psi(\vec{r}) = 0$$

Elektron(en)sin vernachlässigt, Elektronen im 1s-Orbital:

$$\psi(\vec{r}) = \frac{\gamma^{\frac{3}{2}}}{\sqrt{\pi}} e^{-\gamma r}$$

$$\gamma = \frac{Zme^2}{\hbar^2} = Z \frac{e^2}{\hbar c} \frac{mc^2}{\hbar c} \quad (= Z\alpha_f \frac{mc^2}{\hbar c})$$

$$\alpha_f = \frac{e^2}{\hbar c} \approx \frac{1}{137,03}$$

$$\text{Sei nun } Z \rightarrow Z+1 : \tilde{V}(r) = -\frac{(Z+1)e^2}{r} = V(r) - \frac{e^2}{r}$$

$$T + V(r) - \frac{e^2}{r} - E_0] \tilde{\phi}_0(\vec{r}) = 0$$

$$V_{eff} = V(r) + \frac{l(l+1)\hbar^2}{r^2} \frac{1}{2m}$$

$$b) \Delta E^{(1)} = \langle \phi_0 | -\frac{e^2}{r} | \phi_0 \rangle$$

$$c) \Delta E^{(1)} = -\frac{\gamma^3}{\pi} e^2 \int_0^\infty dr r^2 \frac{1}{r} e^{-2\gamma r} \underbrace{\int d\Omega_r}_{4\pi}$$

$$= -e^2 4\gamma^3 \int_0^\infty dr r e^{-2\gamma r} = -e^2 4\gamma^3 (-) \frac{\partial}{\partial \gamma} |_{2\gamma} \int_0^\infty dr e^{-\beta r} = -e^2 4\gamma^3 (-) \frac{\partial}{\partial \beta} |_{2\gamma} \frac{1}{\beta} = e^2 4\gamma^3 \frac{1}{4\gamma^2} =$$

$$-\gamma e^2 = -\frac{Zme^4}{\hbar^2}$$

$$\Rightarrow \Delta E^{(1)} = -\alpha_f \hbar c \gamma = -Z\alpha_f^2 mc^2$$

$$\Rightarrow \text{also stärkere Attraktion}$$

11.2 K3

2-Zustands-Wellenpaket im unendlich tiefen Potential

$$\text{Es gilt: } \psi(x, 0) = A\sqrt{a}(\varphi_1(x) + z\varphi_2(x)), \quad z \in \mathbb{R}$$

$$\text{mit } \int_{-a}^a dx |\varphi_i(x)|^2 = 1,$$

$$\varphi_1 = \frac{1}{\sqrt{a}} \cos\left(\frac{\pi}{2} \frac{x}{a}\right) = \varphi_1(-x), \quad \varphi_2 = \frac{1}{a} \sin\left(\pi \frac{x}{a}\right) = -\varphi_2(-x)$$

$$a) \langle \psi | \psi \rangle = A^2 a (1 + z^2) \stackrel{!}{=} 1 \Rightarrow A = \frac{1}{\sqrt{a}} \frac{1}{\sqrt{1+z^2}}$$

$$\text{Also } \psi(x, 0) = \frac{1}{\sqrt{1+z^2}} (\varphi_1(x) + z\varphi_2(x))$$

$$b) \text{Parität: } \psi(-x, 0) = \frac{1}{\sqrt{1+z^2}} (\varphi_1(-x) + z\varphi_2(-x)) = \frac{1}{\sqrt{1+z^2}} (\varphi_1(x) - z\varphi_2(x))$$

\Rightarrow keine gute Parität!

$$c) \psi(x, 0) \stackrel{VONS}{=} \sum_n c_n \varphi_n = \sum_n \langle \varphi_n | \psi \rangle \varphi_n, \quad \omega_n = |c_n|^2 \omega_1 = \frac{1}{1+z^2}, \quad \omega_2 =$$

$$\frac{z^2}{1+z^2} = 1 - \omega_1, \quad \Rightarrow \omega_{n>1} = 0$$

$$c_n = \langle \varphi_n | \psi \rangle = \int_{-a}^a \varphi_n(x) \psi(x) dx$$

$$d) \psi(x, t) = \frac{1}{\sqrt{1+z^2}} (\varphi_1(x) e^{-i\omega_1 t} + z\varphi_2(x) e^{-i\omega_2 t})$$

$$\omega_1 = \frac{\hbar}{2m} \left(\frac{\pi}{2a}\right)^2, \quad \omega_2 = \frac{\hbar}{2m} \left(\frac{\pi}{a}\right)^2$$

$$e) \langle x \rangle = \langle \psi | x | \psi \rangle = \int_{-a}^a dx \rho(x, t) x$$

$$(\Delta E)^2 = \langle H^2 \rangle - \langle H \rangle^2 = \langle \psi | H^2 | \psi \rangle - \langle \psi | H | \psi \rangle^2$$

$$\langle \psi | H | \psi \rangle = \frac{1}{1+z^2} \langle \varphi_1 + z\varphi_2 | H | \varphi_1 + z\varphi_2 \rangle$$

$$= \frac{1}{1+z^2} (\langle \varphi_1 | H | \varphi_1 \rangle + z^2 \langle \varphi_2 | H | \varphi_2 \rangle) = \frac{1}{1+z^2} (E_1 + z^2 E_2)$$

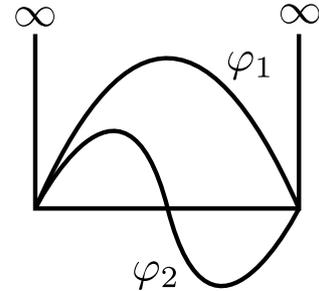
$$(\Delta E)^2 = \langle H^2 \rangle - \langle H \rangle^2 = \frac{1}{1+z^2} (E_1^2 + z^2 E_2^2 - (E_1 + z^2 E_2)^2 \frac{1}{1+z^2})$$

$$= \frac{1}{(1+z^2)^2} (E_1^2(1+z^2-1) + E_2^2(z^2(1+z^2)-z^4) - 2z^2 E_1 E_2)$$

$$E_1 = \frac{\pi^2 \hbar^2}{8ma^2}, \quad E_2 = \frac{\pi^2 \hbar^2}{2ma^2}$$

$$\dots = \left(\frac{z^2}{1+z^2}\right)^2 (E_1^2 + E_2^2 - 2E_1 E_2) = \left(\frac{z^2}{1+z^2}\right)^2 (E_1 - E_2)^2$$

$$= \left(\frac{z^2}{1+z^2}\right)^2 \left(\frac{\hbar^2}{2m} \left(\frac{\pi}{a}\right)^2\right)^2 \left(\left(\frac{1}{2}\right)^2 - 1\right)^2 = \left(\frac{z^2}{1+z^2}\right)^2 \frac{9}{16} \left(\frac{\hbar^2}{2m} \left(\frac{\pi}{a}\right)^2\right)^2$$



p) Zu $\langle p^2 \rangle$:

$$\frac{p^2}{2m} + V(x) = H$$

$$T + V = H$$

$$\langle p^2 \rangle = 2m \langle T \rangle \stackrel{V(x)=0}{=} 2m \langle H \rangle$$

$$\text{oder händisch: } \langle \psi | \left(\frac{\hbar}{i} \frac{\partial}{\partial x}\right)^2 | \psi \rangle$$