7 Übungsblatt von Analysis 4 zum Mittwoch, den 1.6.2011

Aufabe 1

Satz (Blatt 5, Aufgabe 2):

$U \subset \mathbb{R}^n$ offen, $f : \mathbb{R}^n \times U \to \mathbb{R}$ mit

(i) $\partial_2 f$ ex. auf $\mathbb{R}^n \times U$

(ii) $\forall \lambda \in U$ fest ist $f(*, \lambda) \in \mathcal{L}^1$

(iii) $\exists g \in \mathcal{L}^1$, so dass $\forall (x, \lambda) \in \mathbb{R}^n \times U$ gilt $|\partial_2 f(x, \lambda)| \leq g(x)$

$\Rightarrow F : U \to \mathbb{R}, \lambda \mapsto \int\limits_{\mathbb{R}} f(x, \lambda) dx$ diffbar mit

$F'(\lambda) = \int\limits_{\mathbb{R}} \partial_2 f(x, \lambda) dx$

Sei $x_0 \in (0, \infty)$, $\quad \varphi : \underbrace{(x_0, \infty)}_{=:U} \to \mathbb{R}, \quad \varphi(x) := \int\limits_0^\infty \underbrace{e^{-xt}}_{=:f(t,x)} dt = \frac{1}{x}$

$\varphi'(x) = -\frac{1}{x^2}$ zum Satz (i): $\partial_2 f(t, x) = (-t)e^{-xt}$ zu (ii):

$x \in$U fest:

$g_n : \mathbb{R} \to \mathbb{R}, \quad g_n(t) := \mathbb{1}_{[\frac{1}{n}, n]}(t) \underbrace{e^{-xt}}_{st.\ in\ t} \in \mathcal{L}^+ \subset \mathcal{L}^1$ (Beppo Levi)

$g_n \leq g_{n+1}$

$\int\limits_{\mathbb{R}} g_n dt = \int\limits_{\frac{1}{n}}^{n} e^{-xt} dt = [-\frac{1}{x} e^{-xt}]_{\frac{1}{n}}^{n} = \frac{e^{-\frac{x}{n}}}{x} - \frac{e^{-xn}}{x} \underset{n \to \infty}{\longrightarrow} \frac{1}{x}$

$f(*, x) = g_n \underset{n \to \infty}{\longrightarrow} f(*, x) pw., f(*,) \in \mathcal{L}^1$ (Beppo Levi)

zu (iii):

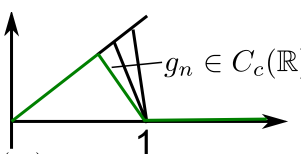
Sei $x \in U, \quad t \in \mathbb{R}$

$|\partial_2 f(x, t)| = | - te^{-xt}| = te^{-xt} \leq \begin{cases} t & , t \in [0, 1] \\ \frac{t}{e^{xt}} & , t \in [1, \infty] \end{cases}$

$\frac{t}{e^{xt}} = \frac{t}{\sum\limits_{n=0}^{\infty} \frac{(xt)^n}{n!}} \leq \frac{t}{\frac{(xt)^3}{3!}} = \frac{6}{x^3} \frac{1}{t^2} \leq \frac{6}{x_0^3} \frac{1}{t^2}$

$\Rightarrow |\partial_2 f(t, x)| \leq \underbrace{t \mathbb{1}_{[0,1]}(+)}_{\in \mathcal{L}^+ \subset \mathcal{L}^1} + \underbrace{\frac{6}{x_0^3} \frac{1}{t^2} \mathbb{1}_{[1,\infty]}(+)}_{\in \mathcal{L}^+ \subset \mathcal{L}^1} =: g(t)$

$\int g_n \leq \int\limits_0^1 t dt = [\frac{1}{2} t^2]_0^1 = \frac{1}{2} < \infty$



(i)-(iii) gelten mit $g : \mathbb{R} \to \mathbb{R}, \quad g(t) := t \mathbb{1}_{[0,1]}(+) + \frac{6}{x_0^3} \frac{1}{t^2} \mathbb{1}_{[1,\infty]}(+)$

$\Rightarrow \phi'(x) = \int\limits_0^\infty \partial_2 f(t, x) dt = \int\limits_0^\infty (-te^{-xt}) dt = - \int\limits_0^\infty te^{-xt} dt$

$\Rightarrow \varphi^{(n)}(x) = (-1) \frac{1}{x^{n-1}} n!$

(i): ...$(-t) \to (-1)^n$

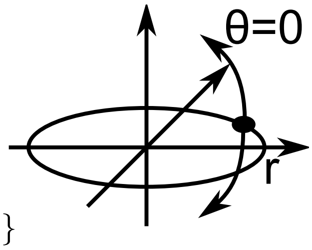
(ii) gilt aus (iii) im Schritt n-1

(iii):$\partial_2 \to \partial_2^n$ und $-t \to (-t)^n$ bzw. $t \to t^n$

$\Rightarrow \varphi^{(n)}(x) = \int\limits_0^\infty \partial_2^n f(t, x) dx = (-1)^n \int\limits_0^\infty t^n e^{-xt} dt$

$\Gamma(n + 1) \int\limits_0^\infty t^n e^{-t} dt = (-1)^n \varphi^{(n)}(1) = (-1)^n (-1)^n \frac{n!}{1^{n+1}} = n!$

Aufabe 2



$\phi : (0,\infty) \times (0,\pi) \times (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}^3$

$\phi(r,\varphi,\theta) = (r\cos(\varphi)\cos(\theta), r\sin(\varphi)\cos(\vartheta), r\sin(\theta))$

$Bild(\phi) = \mathbb{R}^3 \setminus \{x \in \mathbb{R}^3 \big| x_1 \geq 0,\ x_2 = 0\} = \mathbb{R}^3 \setminus \mathbb{R}^2 \times \{0\}$

$\mathbb{R}^2 \times \{0\}$ echter Unterram von $\mathbb{R}^3$ und damit eine Nullmenge

$$\Rightarrow \int\limits_{B_{||.||_2}(0,1)} \frac{1}{||x||_2^\alpha} dx = \int\limits_{B_{||.||}(0,1)\setminus \mathbb{R}\times\{0\}\times\mathbb{R}} \frac{1}{||x||_2^\alpha} dx$$

$$B_{||.||}(0,1) \setminus \mathbb{R} \times \{0\} \times \mathbb{R} = Bild(\varphi|_{(0,1)\times(0,2\pi)\times(-\frac{\pi}{2},\frac{\pi}{2})})$$

$$... = \int\limits_{(0,1)\times(0,2\pi)\times(-\frac{\pi}{2},\frac{\pi}{2})} \frac{1}{||\phi(x)||_2^\alpha} \cdot \det(D(\varphi(x)))|dx$$

$$= \int\limits_0^1 \int\limits_0^{2\pi} \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{r^\alpha} \cdot r^2 \cos(\theta) d\theta d\varphi dr$$

$$= 4\pi \int\limits_0^1 r^{2-\alpha} = 4\pi \left[\frac{1}{3-\alpha} r^{3-\alpha}\right]_0^1$$

$$= \frac{4\pi}{3-\alpha}(1 - 0^{3-\alpha})$$

(geht da $3 - \alpha > 0$)

Aufabe 3

$\varepsilon > 0, \quad \alpha_\varepsilon : \mathbb{R} \to \mathbb{R},\ \alpha_\varepsilon(x) := \tanh(\frac{x}{\varepsilon})$

$\sinh(x) = \frac{e^x - e^{-x}}{2}, \quad \cosh(x) = \frac{e^x + e^{-x}}{2}$

$\Rightarrow \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$\sinh'(x) = \cosh(x), \quad \cosh'(x) = \sinh(x)$

$\Rightarrow \tanh'(x) = \frac{1}{\cosh^2(x)}$

$$\int\limits_{-1}^1 f(x) d\alpha_\varepsilon = \int\limits_{-1}^1 f(x)\alpha'_\varepsilon(x) dx = \int\limits_{-1}^1 f(x) \frac{1}{\varepsilon} \frac{1}{\cosh^2(\frac{x}{\varepsilon})} dx$$

$$= \int\limits_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} f(x\varepsilon) \frac{1}{\cosh^2(x)} dx = \int\limits_{\mathbb{R}} \underbrace{\underbrace{\mathbb{1}_{[-\frac{1}{\varepsilon},\frac{1}{\varepsilon}]}(x) f(x\varepsilon)}_{\underset{\varepsilon \to 0}{\longrightarrow} f(0)\ pw} \frac{1}{\cosh^2(x)}}_{\underset{\varepsilon \to 0}{\longrightarrow} \frac{f(0)}{\cosh^2(x)}\ pw} dx$$

$$|\mathbb{1}_{[-\frac{1}{\varepsilon},\frac{1}{\varepsilon}]}(x) f(x\varepsilon) \frac{1}{\cosh^2(x)}| \leq \frac{||f||_\infty}{\cosh^2(x)}$$

$$g_n : \mathbb{R} \to \mathbb{R}, \quad g_n(x) := \mathbb{1}_{[-n,n]}(x) \frac{1}{\cosh^2(x)} \in \mathcal{L}^+ \subset \mathcal{L}^1$$

$$\int\limits_{\mathbb{R}} g_n(x) dx = \int\limits_{-n}^n \frac{1}{\cosh^2(x)} dx = [\tanh(x)]_{-n}^n$$

$$= \tanh(n) - \tanh(-n) \underset{n \to \infty}{\longrightarrow} 1 - (-1) = 2$$

Beppo Levi $\Rightarrow \frac{1}{\cosh^2(x)} \in \mathcal{L}^1 mit \int\limits_{\mathbb{R}} \frac{1}{\cosh^2(x)} dx = 2 \quad (= \lim\limits_{n\to\infty} \int\limits_{\mathbb{R}} g_n dx)$

$\Rightarrow \frac{||f||_\infty}{\cosh^2(x)} \in \mathcal{L}^1$

pw. $\frac{f(0)}{\cosh^2(x)} \overset{\varepsilon \to 0}{\longleftarrow} \mathbb{1}_{[-\frac{1}{\varepsilon},\frac{1}{\varepsilon}]}(x) f(x\varepsilon) \frac{1}{\cosh^2(x)}, \quad |...| \leq \frac{||f||_\infty}{\cosh^2(x)} \in \mathcal{L}^1$

Lebesgue:

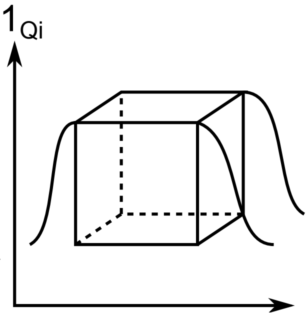
$$\Rightarrow \int\limits_{\mathbb{R}} \mathbb{1}_{[-\frac{1}{\varepsilon},\frac{1}{\varepsilon}]}(x) f(x\varepsilon) \frac{1}{\cosh^2(x)} dx \underset{\varepsilon \to 0}{\longrightarrow} \int\limits_{\mathbb{R}} \frac{f(0)}{\cosh^2(x)} dx = f(0) \int\limits_{\mathbb{R}} \frac{1}{\cosh^2(x)} dx = 2f(0)$$

Aufabe 4

$f : \mathbb{R} \to \mathbb{R},\ \ f(x) := \mathbb{1}_{(0,\infty)}(x)\frac{1}{\sqrt{x}+x^4}$ $g_1 : \mathbb{R} \to \mathbb{R},\ \ x \mapsto \mathbb{1}_{(0,1]}(x)\frac{1}{\sqrt{x}} \in \mathcal{L}^+ \subset \mathcal{L}^1$
$g_2 : \mathbb{R} \to \mathbb{R},\ \ x \mapsto \mathbb{1}_{[1,\infty)}(x)\frac{1}{x^4} \in \mathcal{L}^+ \subset \mathcal{L}^1$
Sei $g := g_1 + g_2$. Dann ist $g \in \mathcal{L}^1$ mit:
Für $x \in \mathbb{R}$ gilt:

$$f(x) = \underbrace{\mathbb{1}_{(0,1]}(x)\underbrace{\frac{1}{\sqrt{x}+x^4}}_{\le \frac{1}{\sqrt{x}}}}_{\le g_1(x)} + \underbrace{\mathbb{1}_{(1,\infty)}(x)\underbrace{\frac{1}{\sqrt{x}+x^4}}_{\le \frac{1}{x^4}}}_{\le g_2} \le g_1(x) + g_2(x) = g(x)$$

wegen $f(x) \ge 0 \forall x \in \mathbb{R}$ ist damit auch $|f(x)| \le g(x)$
Weiter ist f als stetige Funktion messbar.
$\Rightarrow f$ messbar, $|f| \le g \in \mathcal{L}^1 \quad \Rightarrow f \in \mathcal{L}^1$



Aufabe 5

Ω offen, beschr. $\Rightarrow \Omega$ mbar:
Ω offen $\Rightarrow \exists (Q_i)$ mit $\Omega = \bigcup\limits_{i\in\mathbb{N}} Q_i,\quad Q_i \cap Q_j \subset \partial Q_i \forall i \ne j$
$\mathbb{1}_\Omega = \mathbb{1}_{\bigcup\limits_{i\in\mathbb{N}} Q_i}$
$\int \mathbb{1}_{\bigcup\limits_{i\in\mathbb{N}} Q_i} = \sum\limits_{i\in\mathbb{N}} \int \mathbb{1}_{Q_i}$
$\mathbb{1}_{Q_i} \in \mathcal{L}^+$
$\sum\limits_{i=1}^{N} \mathbb{1}_{Q_i} \in \mathcal{L}^+$. Genauso $\sum\limits_{i=1}^{N} \mathbb{1}_{int(Q_i)} \in \mathcal{L}^1$
$\Rightarrow \frac{1}{2}\sum\limits_{i=1}^{N}(\mathbb{1}_{Q_i} + \mathbb{1}_{\mathring{Q}_i}) \in \mathcal{L}^1$
$\underset{N\to\infty}{\longrightarrow} \frac{1}{2}\sum\limits_{i\in\mathbb{N}}(\mathbb{1}_{Q_i} + \mathbb{1}_{\mathring{Q}_i}) = \mathbb{1}_{\bigcup\limits_{i\in\mathbb{N}} Q_i} = \mathbb{1}_\Omega \in \mathcal{L}^+$
insb. Ω mbar!

a) f beschr. mbar. $\Rightarrow f\mathbb{1}_\Omega$ mbar, $|f\mathbb{1}_\Omega| \le ||f||_\infty \mathbb{1}_\Omega \in \mathcal{L}^1$
   $\mathbb{1}_\Omega \in \mathcal{L}^1$ da mbar und endl, also $\int\limits_{\mathbb{R}} \mathbb{1}_\Omega < \infty$

b) $x \in \Omega,\ f^*(x) = \sup\{f(\tau^j(x)) | j \in \mathbb{N}\} = \lim\limits_{N\to\infty} \underbrace{\max\limits_{j\in\{1,\ldots,N\}} \underbrace{f(\tau^j(x))}_{mbar,\ beschr.}}_{mbar,\ beschr.\ d\ ||f||_\infty \mathbb{1}_\Omega \Rightarrow \in \mathcal{L}^1}$

   $\overset{(Lebesgue)}{\Rightarrow} f^* \in \mathcal{L}^1$