

$\sin^4(x)$  Fourierreihe?

$$\sum_{k \in \mathbb{Z}} \langle e^{ikx} | f \rangle e^{ikx} \cdot \frac{1}{2\pi}$$

$$\sum \frac{1}{2} (\langle \sin(hx) | f \rangle \sin(hx) + \langle \cos(hx) | f \rangle \cos(hx)) + \langle \frac{1}{\sqrt{2}} | f \rangle \frac{1}{\sqrt{2}}$$

$$\sum a_n z^n \quad a_n = \frac{1}{2\pi i} \oint \frac{f(z)}{z^{n+1}} dz$$

$$\sin^4(x) = \left( \frac{e^{ix} - e^{-ix}}{2i} \right)^4 = \frac{1}{(2i)^4} (e^{4ix} - 4e^{3ix}e^{-ix} + 6e^{2ix}e^{-2ix} - 4e^{ix}e^{-3ix} + e^{-4ix})$$

$$\sin^4(x) = \frac{1}{16} (e^{4ix} - 4e^{2ix} + 6 - 4e^{-2ix} + e^{-4ix})$$

$$= \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

$$= \frac{1}{16} (2 \cos(4x) - 4 \cdot 2 \cos(2x) + 6)$$

$$= \frac{6}{16} \cdot 1 - \frac{4 \cdot 2}{16} \cos(2x) + \frac{2}{16} \cos(4x)$$

$$\langle \sin^4(x) | \sin(hx) \rangle$$

$$\frac{1}{2} \langle \cos(4x) | \sin^4(x) \rangle$$

$$\sum a_n \varphi_n$$

$\langle \varphi_n | f \rangle$  orthogonal!

$$[0, 4\pi] e^{i \frac{k}{2} x}$$

$$\frac{1}{x^2 \sin^2(x) + e^x} \mathbb{1}_{[0, \infty)} \text{ messbar}$$

$f$  messbar  $\Leftrightarrow f$  stetig, Ableitung von differenzierbarer Funktion,  $f$  monoton, Indikator von messbarer Menge

$$\int \underbrace{\frac{1}{x^2 \sin^2(x) + e^x}}_{\geq 0} \mathbb{1}_{[0, \infty)} dx \leq \int_0^{\infty} e^{-x} dx = 1 < \infty$$

$$\leq \frac{1}{e^2} \mathbb{1}_{[0, \infty)}$$

$$\| \frac{1}{x^2 \sin^2(x) + e^x} \|_{\infty} \leq \| e^{-x} \|_{\infty} = 1 < \infty$$

$$\text{ess-sup } \mathbb{1}_Q = 0 \quad \mathbb{1}_Q = 0 \quad \text{f.ü.}$$

$$\text{ess-sup } f = \inf_{\lambda > 0} \sup_{x \in \mathbb{R}} f(x)$$

$$\lambda(\underbrace{\{x \in \mathbb{R} \mid \mathbb{1}_Q(x) = 1\}}_{\mathbb{1}_Q^{-1}(1)}) = \lambda(Q) = 0$$

$$\lambda(\mathbb{1}_Q^{-1}\{0\}) = \lambda(\mathbb{R} \setminus Q) = \lambda(\mathbb{R}) - \lambda(Q) = \infty - 0 = \infty$$

$$\text{sup}(\mathbb{1}_Q) = 1 \quad \|\mathbb{1}_Q\|_{\infty}^{\text{ess-sup}} = 0$$

$$f(x) = \begin{cases} 0 & \forall x \in \mathbb{R} \setminus \{0\} \\ \infty & x=0 \end{cases} = 0 \quad \text{f.ü.}$$

$$\sin x \in \mathcal{L}_1(\mathbb{R}) ? \quad \text{Nein, kein integrierbares } \sin(x) \in \mathcal{L}_1^+$$

$$\Rightarrow \cos(x) = \sin(x - \frac{\pi}{2}) \in \mathcal{L}_1^+$$

$$\Rightarrow e^{ix} = \cos(x) + i \sin(x) \in \mathcal{L}_1^+$$

$$\int |e^{ix}| dx = \infty \quad \text{⚡}$$

$$\sin x \geq \frac{1}{2} \quad \forall x \in [\frac{\pi}{6}, \frac{5}{6}\pi]$$

$$\int |\sin x| dx \geq \sum_{h \in \mathbb{R}} \int_{[\frac{\pi}{6} + h, \frac{5}{6}\pi + h]} |\sin x| dx = \infty$$

$\underbrace{\qquad\qquad\qquad}_{\geq \frac{1}{2}}$

$$\log|x| e^{-|x|} \quad x=0 \text{ Problem?} \quad \text{Nein, denn:}$$

Positiv:  
okay!

$$\int |\log|x|| e^{-|x|} dx = 2 \int_0^1 \underbrace{|\log|x|| e^{-|x|}}_{\leq 1} dx + 2 \int_1^{\infty} \underbrace{|\log|x|| e^{-|x|}}_{\leq x} dx$$

$\int \frac{1}{x} dx$  Pfrü!

$$\leq \int_0^1 |\log|x|| dx + \underbrace{\qquad\qquad\qquad}_{< \infty}$$

$$\stackrel{x \rightarrow 0}{\rightarrow} \lim_{x \rightarrow 0} (x \log x - x) \Big|_0^1 = 1 < \infty$$

$\mathcal{L}^1, \mathcal{L}^2, \mathcal{L}^\infty$

$f \in \mathcal{L}^\infty: \exists c > 0 \mid |f| \leq c$  f.ü.

$\frac{1}{x} \notin \mathcal{L}^\infty(\mathbb{R})$ , denn  $\forall c > 0: \mid \frac{1}{x} \mid \geq c \quad \forall x \in [-\frac{1}{c}, \frac{1}{c}]$   
 $\lambda(\dots) = \frac{2}{c} > 0$

$\frac{1}{x} \mathbb{1}_{[-\varepsilon, \varepsilon]^c} \in \mathcal{L}^\infty$

$$\|f\|_1 = \underbrace{\int_{-\infty}^{-\varepsilon} \left| \frac{1}{x} \right| dx}_{=\infty} + \underbrace{\int_{\varepsilon}^{\infty} \left| \frac{1}{x} \right| dx}_{=\infty} = \infty \Rightarrow f \notin \mathcal{L}^1$$

$\frac{1}{x+x^3} \notin \mathcal{L}^1 \quad \frac{1}{x+x^3} \mathbb{1}_{(1, \infty)} \in \mathcal{L}^1$

$$\int_0^1 \frac{1}{x+x^3} \geq \frac{1}{2} \int_0^1 \frac{1}{x} dx = \infty$$

$\frac{1}{\sqrt{x}} \mathbb{1}_{[0,1]} \in \mathcal{L}^1 \notin \mathcal{L}^2 \Rightarrow$  Singularitäten am Null!!!

$$f_n \xrightarrow{\text{f.ü.}} f \stackrel{!}{\Rightarrow} \int f_n \rightarrow \int f$$

falls  $|f_n| \leq g \in \mathcal{L}^1$

$$\underbrace{e^{-\frac{|x|}{n}} \mathbb{1}_{[0,1]}}_{\xrightarrow{n \rightarrow \infty} 1} \quad |e^{-\frac{|x|}{n}}| \leq 1 \in \mathcal{L}^1[0,1]$$

$$\Rightarrow \int_0^1 e^{-\frac{|x|}{n}} \rightarrow \int_0^1 1 = 1$$

BEIPPELE LEVI

$$\int_0^\infty e^{-x/n} dx \xrightarrow{n \rightarrow \infty} \int_0^\infty 1 dx = \infty$$
$$\int_0^\infty e^{-x} dx = 1$$

$$\int_{\mathbb{R}} \cos\left(\frac{x}{n}\right) \cdot \frac{1}{x^2+1} dx \xrightarrow{\text{Lebesgue}} \int_{\mathbb{R}} 1 \cdot \frac{1}{x^2+1} dx = \pi$$

$\xrightarrow{n \rightarrow \infty} 1$

$$\underbrace{|\cos\left(\frac{x}{n}\right) \cdot \frac{1}{x^2+1}|}_{|1| \leq 1} \leq \frac{1}{x^2+1} \in \mathcal{L}^1 \quad \left( \int_{\mathbb{R}} \frac{1}{x^2+1} \leq \int_{\mathbb{R}} 1 + \int_1^{\infty} \frac{1}{x^2} dx + \int_{-\infty}^{-1} \frac{1}{x^2} dx < \infty \right)$$

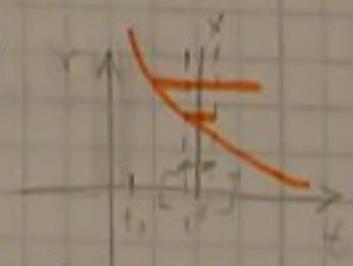
$$\int_{\mathbb{R}} \frac{dx}{x^2} \geq \int_0^1 \frac{dx}{x^2} = -\frac{1}{x} \Big|_0^1 \quad \Downarrow \quad \frac{1}{x} \notin \mathcal{L}^2(\mathbb{R})$$

$$\neg \exists \int_{\mathbb{R}} \frac{1}{x^2} dx !$$

$$\partial_x \int \cos(xy) \cdot \frac{1}{y^4+1} dy = \int \partial_x \cos(xy) \frac{1}{y^4+1} dy = - \int \sin(xy) \frac{y}{y^4+1} dy$$

$\in C^1 \mid \partial_x \cos(xy) = -y \sin(xy)$

$$\left| \frac{y \sin(xy)}{y^4+1} \right| \leq \frac{y}{y^4+1} \in \mathcal{L}^1 \quad \text{unabhängig von } x!$$



$\forall t \geq t_0 > 0:$

$$\partial_t \int_1^{\infty} \frac{e^{-xt}}{x} dx \quad \left| \partial_t \frac{e^{-xt}}{x} \right| = | -e^{-xt} | \leq e^{-t_0 x} \in \mathcal{L}^1$$

$$= \int_1^{\infty} \partial_t \frac{e^{-xt}}{x} dx = \int_1^{\infty} -e^{-xt} dx = \frac{1}{t} e^{-xt} \Big|_1^{\infty} = -\frac{1}{t} e^{-t}$$

Plancherel:  $\|\hat{f}\|_2 = \|f\|_2$

$$\hat{f}(x) = \frac{1}{\sqrt{2\pi}} \int f(t) e^{-ixt} dt \quad \forall f \in \mathcal{L}^1$$

$$\forall f \in \mathcal{L}^2: \underbrace{f \mathbb{1}_{[-a, a]}}_{\in \mathcal{L}^1} \in \mathcal{L}^1 \quad \|fg\|_1 \leq \|f\|_2 \|g\|_2 !$$

$$\Rightarrow \underbrace{f \mathbb{1}_{[-a, a]}}_{\in \mathcal{L}^1} \text{ hat Fouriertransformierte } \widehat{f \mathbb{1}_{[-a, a]}} \\ \underbrace{f \mathbb{1}_{[-a, a]}}_{\in \mathcal{L}^2} \xrightarrow{f.i.} f$$

$$|f|^2 \mathbb{1}_{[-n,n]} \leq |f|^2 \in \mathcal{L}^1 \quad (f \in \mathcal{L}^2)$$

$$\Rightarrow \underbrace{f \mathbb{1}_{[-n,n]}}_{f_n} \xrightarrow{\mathcal{L}^2} f$$

$$\Leftrightarrow \|\hat{f}_n - \hat{f}\|_2 = \|\widehat{f_n - f}\|_2 = \|f_n - f\|_2 \rightarrow 0$$

⌋  
Konvergenz in  $\mathcal{L}^2$  durch Satz von Plancherel

$$\frac{1}{\sqrt{2\pi}} \int_{-n}^n f(t) e^{-ixt} dt \xrightarrow[n \rightarrow \infty]{\mathcal{L}^2} \hat{f}$$