1 Präsenzaufgabe 18.10.12

Erinnerung

Kommutator:

$$[A+B,C+D]=[A+B,C]+[A+B,D]=[A,C]+[B,C]$$

Operatoren:

$$x=\sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger)\ ,\qquad p=-i\sqrt{\frac{\hbar m\omega}{2}}(a-a^\dagger)\ ,\qquad <a|b>=\delta_{ab}$$

a) $[x,p]\psi(t)=(xp-px)\psi(t)=-xi\hbar\frac{\partial}{\partial x}\psi(t)+i\hbar\frac{\partial x}{\partial x}\psi(t)=i\hbar\psi(t)$

$$\begin{aligned}&[a,a^\dagger]\psi(t)=aa^\dagger\psi(t)-a^\dagger a\psi(t)\\&=\tfrac{1}{2\hbar m\omega}(m\omega x+ip)(m\omega x-ip)\psi(t)-\tfrac{1}{2\hbar m\omega}(m\omega x-ip)(m\omega x+ip)\psi(t)\\&=\tfrac{1}{2\hbar m\omega}(((m\omega x)^2-m\omega xip+ipm\omega x+pp)\psi(t)\\&\quad-((m\omega x)^2+m\omega xip-ipm\omega x+pp)\psi(t))\\&=\tfrac{1}{2\hbar m\omega}(-2m\omega xip+2ipm\omega x)\psi(t)=\psi(t)\end{aligned}$$

b) $(aa^\dagger+a^\dagger a)\psi(t)$

$$\begin{aligned}&=\tfrac{1}{2\hbar m\omega}(((m\omega x)^2-m\omega xip+ipm\omega x+pp)\psi(t)\\&\quad+((m\omega x)^2+m\omega xip-ipm\omega x+pp)\psi(t))\\&=\tfrac{1}{\hbar m\omega}(((m\omega x)^2+p^2)\psi(t))\\&=\tfrac{1}{2\hbar m\omega}(((m\omega x)^2-m\omega xip+ipm\omega x+pp)\psi(t)\\&\quad+((m\omega x)^2+m\omega xip-ipm\omega x+pp)\psi(t))\\&H=\tfrac{p^2}{2m}+\tfrac{m}{2}\omega^2x^2=\tfrac{1}{2m}\tfrac{\hbar m\omega}{\hbar m\omega}p^2+\tfrac{\hbar m\omega}{\hbar m\omega}\tfrac{1}{2m}m^2\omega^2x^2\\&\Rightarrow H\psi(t)=\tfrac{1}{2}\omega\hbar(aa^\dagger+a^\dagger a)\end{aligned}$$

c) Ansatz: $\frac{a^{\dagger m}|0>}{\sqrt{m!}}$

$$\begin{aligned}&m=0: &&\frac{a^{\dagger 0}|0>}{\sqrt{0!}}=1|0>\\&m=m+1: &&\frac{a^{\dagger m+1}|0>}{\sqrt{(m+1)!}}=\frac{\sqrt{(n+1)!}}{\sqrt{(n+1)!}}|m+1>\end{aligned}$$

d) $H|n>=\frac{1}{2}\omega\hbar(aa^\dagger+a^\dagger a)|n>=\frac{1}{2}\omega\hbar(a\sqrt{n+1}|n+1>+a^\dagger\sqrt{n}|n-1>)$

$$\begin{aligned}&=\tfrac{1}{2}\omega\hbar(\sqrt{n+1}^2|n>+\sqrt{n}^2|n>)\\&=\underbrace{\frac{1}{2}\omega\hbar(2n+1)}_{EnergieEW}|n>\end{aligned}$$

e) $(\Delta x)^2=<x^2>-<x>^2=<0|\frac{\hbar}{2m\omega}(a+a^\dagger)^2|0>-(<0|\sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger)|0>)^2$

$$=\frac{\hbar}{2m\omega}<0|\sqrt{1}^2|0>=\frac{\hbar}{2m\omega}$$