$\boxed{H5\,(a)}$

- Ruhemasse von zwei Deuteriumkernen vor der Reaktion

  $m_0(\text{vorher}) = 2\, m_0(D) = 4.0294 \text{ amu}$

- Ruhemasse nach der Reaktion

  $m_0(\text{nachher}) = m_0(He) = 4.0039 \text{ amu}$

$\Rightarrow$ Massenverlust von $\Delta m = m_0(\text{vorher}) - m_0(\text{nachher})$

$= 0.0255 \text{ amu}$

$\Rightarrow$ daraus folgt für die freiwerdende Energie mithilfe von $E = mc^2$: $\Delta E = \Delta m \cdot c^2$

also: pro Deuterium-Masse wird die Energie

$$\frac{\Delta E}{2m_0(D)} = \frac{\Delta m}{2m_0(D)} \cdot c^2 = 0.00633\, c^2$$

erzeugt. Damit ergibt sich für die benötigte Menge Deuterium (Masse M) für eine bestimmte Energie E

$$M(D) = \frac{E}{c^2} \cdot \frac{1}{0.00633}$$

Bei einem jährlichen Energieverbrauch von $14 \cdot 10^{18}$ J würde man also

$$M(D) = \frac{14 \cdot 10^{18}}{(3 \cdot 10^8)^2} \cdot \frac{1}{0.00633} \quad \frac{kg\, m^2}{s^2} \; \frac{s^2}{m^2}$$

$$= 25 \cdot 10^3 \text{ kg}$$

Deuterium brauchen.

H5 (b)

Energieabstrahlung gleichmäßig nach allen Seiten

$\rightarrow$ Kugelfläche um die Sonne im Abstand Erde - Sonne

$$A = 4\pi R_{ES}^2 = 2.83 \cdot 10^{23}\ m^2$$

Energieabstrahlung in $\Delta t = 1\ sec$:

$$\Delta E = \varepsilon \cdot A \cdot \Delta t$$

$$= 1.4 \cdot 10^3 \cdot 2.83 \cdot 10^{23} \cdot 1\ \frac{J}{m^2 s}\ m^2 s \qquad (1W = 1\ J/s)$$

$$= 3.96 \cdot 10^{26}\ J$$

daraus folgt für den Massenverlust pro sec:

$$\Delta m = \frac{\Delta E}{c^2} = 4.4 \cdot 10^9\ kg \qquad (c = 3 \cdot 10^8\ m/s)$$

Lebensdauer der Sonne:

$$T = \Delta t\ \frac{M_S}{\Delta m} = \frac{1 \cdot 1.99 \cdot 10^{30}}{4.4 \cdot 10^9}\ s = 4.5 \cdot 10^{20}\ s$$

$$\sim 1.4 \cdot 10^{13}\ \text{Jahre}$$

Dieser Wert ist nicht realistisch, da nur ein Bruchteil der Masse überhaupt zerstrahlen kann (etwa 1/1000).

$\rightarrow$ dann $T \sim 10^{10}$ Jahre

H6]

Lösung: The equations of motions read (Newton's second law)

$$m_1\ddot{\boldsymbol{r}}_1 = \boldsymbol{F}_1 = -\frac{\partial V_{12}(|\boldsymbol{r}_1-\boldsymbol{r}_2|)}{\partial \boldsymbol{r}_1},$$
$$m_2\ddot{\boldsymbol{r}}_2 = \boldsymbol{F}_2 = -\frac{\partial V_{12}(|\boldsymbol{r}_1-\boldsymbol{r}_2|)}{\partial \boldsymbol{r}_2}.$$

Verify that the sum of all forces equals zero. Denote $r \equiv |\boldsymbol{r}_1 - \boldsymbol{r}_2|$ and $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$

$$\boldsymbol{F}_1+\boldsymbol{F}_2 = -\frac{\partial V_{12}(r)}{\partial r}\frac{\partial r}{\partial \boldsymbol{r}_1} - \frac{\partial V_{12}(r)}{\partial r}\frac{\partial r}{\partial \boldsymbol{r}_2} = -\frac{\partial V_{12}(r)}{\partial r}\left(\frac{\partial r}{\partial \boldsymbol{r}_1}+\frac{\partial r}{\partial \boldsymbol{r}_2}\right),$$

$$\frac{\partial r}{\partial x_1} = \frac{\partial\sqrt{(x_1-x_2)^2+\dots}}{\partial x_1} = \frac{2(x_1-x_2)}{2\sqrt{(x_1-x_2)^2+\dots}},$$

$$\frac{\partial r}{\partial x_2} = \frac{\partial\sqrt{(x_1-x_2)^2+\dots}}{\partial x_1} = -\frac{2(x_1-x_2)}{2\sqrt{(x_1-x_2)^2+\dots}},$$

$$\frac{\partial r}{\partial \boldsymbol{r}_1} = \frac{\boldsymbol{r}}{r}, \qquad \frac{\partial r}{\partial \boldsymbol{r}_2} = -\frac{\boldsymbol{r}}{r}$$

$$\boldsymbol{F}_1+\boldsymbol{F}_2 = -\frac{\partial V_{12}(r)}{\partial r}\left(\frac{\boldsymbol{r}}{r}-\frac{\boldsymbol{r}}{r}\right) = 0.$$

Verify that the sum of momenta of all forces equals zero

$$\boldsymbol{r}_1\times\boldsymbol{F}_1+\boldsymbol{r}_2\times\boldsymbol{F}_2 = -\frac{\partial V_{12}(r)}{\partial r}\boldsymbol{r}_1\times\frac{\boldsymbol{r}}{r}+\frac{\partial V_{12}(r)}{\partial r}\boldsymbol{r}_2\times\frac{\boldsymbol{r}}{r} = -\frac{\partial V_{12}(r)}{\partial r}\frac{\boldsymbol{r}\times\boldsymbol{r}}{r} = 0.$$

The system is closed! It has invariants associated with all continuous symmetry transformations, such as translational invariance of time (total energy is conserved), translational invariance of space (total momentum is conserved), rotational invariance of space (total angular momentum is conserved).

Now consider the total energy

$$E = \frac{m_1\dot{\boldsymbol{r}}_1^2}{2}+\frac{m_2\dot{\boldsymbol{r}}_2^2}{2}+V_{12}(|\boldsymbol{r}_1-\boldsymbol{r}_2|).$$

The rate at which E changes in time is given by

$$\frac{dE}{dt} = m_1\dot{\boldsymbol{r}}_1\ddot{\boldsymbol{r}}_1+m_2\dot{\boldsymbol{r}}_2\ddot{\boldsymbol{r}}_2+\frac{\partial V_{12}(|\boldsymbol{r}_1-\boldsymbol{r}_2|)}{\partial \boldsymbol{r}_1}\dot{\boldsymbol{r}}_1+\frac{\partial V_{12}(|\boldsymbol{r}_1-\boldsymbol{r}_2|)}{\partial \boldsymbol{r}_2}\dot{\boldsymbol{r}}_2,$$

$$\frac{dE}{dt} = \dot{\boldsymbol{r}}_1\left(m_1\ddot{\boldsymbol{r}}_1-\boldsymbol{F}_1\right)+\dot{\boldsymbol{r}}_2\left(m_2\ddot{\boldsymbol{r}}_2-\boldsymbol{F}_2\right) = 0.$$

Thus the total energy is conserved! This result follows from the fact that the potential energy does not depend explicitly on time.

Analogously, for the total momentum

$$\boldsymbol{P} = m_1\dot{\boldsymbol{r}}_1+m_2\dot{\boldsymbol{r}}_2,$$

$$\frac{d\boldsymbol{P}}{dt} = m_1\ddot{\boldsymbol{r}}_1+m_2\ddot{\boldsymbol{r}}_2 = \boldsymbol{F}_1+\boldsymbol{F}_2 = 0,$$

Thus the total momentum is conserved! This result follows from the vanishing of the sum of all forces, which is always the case when the potential energy is invariant with respect to translations of coordinates ($\boldsymbol{r}_i \to \boldsymbol{r}_i+\boldsymbol{a}$).

For the total angular momentum relative to the center of mass, we have

$$\boldsymbol{L}_r = m_1(\boldsymbol{r}_1-\boldsymbol{R})\times\left(\dot{\boldsymbol{r}}_1-\dot{\boldsymbol{R}}\right)+m_2(\boldsymbol{r}_2-\boldsymbol{R})\times\left(\dot{\boldsymbol{r}}_2-\dot{\boldsymbol{R}}\right) = \frac{m_1m_2}{m_1+m_2}\boldsymbol{r}\times\dot{\boldsymbol{r}},$$

where

$$\boldsymbol{R} = \frac{m_1\boldsymbol{r}_1+m_2\boldsymbol{r}_2}{m_1+m_2},$$
$$\boldsymbol{r} = \boldsymbol{r}_1-\boldsymbol{r}_2.$$

Therefore,

$$\frac{dl}{dt} = \frac{m_1 m_2}{m_1 + m_2}(\dot{\boldsymbol{r}} \times \dot{\boldsymbol{r}} + \boldsymbol{r} \times \ddot{\boldsymbol{r}}) = \frac{m_1 m_2}{m_1 + m_2}\boldsymbol{r} \times \ddot{\boldsymbol{r}},$$

$$\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{r}}_1 - \ddot{\boldsymbol{r}}_2 = \frac{\boldsymbol{F}_1}{m_1} - \frac{\boldsymbol{F}_2}{m_2},$$

$$\boldsymbol{F}_1 = -\frac{\partial V_{12}(r)}{\partial r}\frac{\partial r}{\partial \boldsymbol{r}_1} = -\frac{\partial V_{12}(r)}{\partial r}\frac{\boldsymbol{r}}{r},$$

$$\boldsymbol{F}_2 = -\frac{\partial V_{12}(r)}{\partial r}\frac{\partial r}{\partial \boldsymbol{r}_2} = \frac{\partial V_{12}(r)}{\partial r}\frac{\boldsymbol{r}}{r},$$

$$\ddot{\boldsymbol{r}} = -\frac{\partial V_{12}(r)}{\partial r}\frac{m_1 + m_2}{m_1 m_2}\frac{\boldsymbol{r}}{r},$$

$$\boldsymbol{r} \times \ddot{\boldsymbol{r}} = -\frac{\partial V_{12}(r)}{\partial r}\frac{m_1 + m_2}{m_1 m_2}\frac{\boldsymbol{r} \times \boldsymbol{r}}{r} = 0,$$

$$\frac{d\boldsymbol{L}_r}{dt} = 0.$$

Thus the total angular momentum relative to the center of mass is conserved! We have used explicitly that the potential energy depends only on $r = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$. More generally, the angular momentum relative to the center of mass is conserved whenever two conditions are fulfilled: (i) sum of all forces equals zero and (ii) sum of momenta of all forces equals zero.

Note that the total angular momentum of two particles can be written as follows

$$\boldsymbol{L} = m_1 \boldsymbol{r}_1 \times \dot{\boldsymbol{r}}_1 + m_2 \boldsymbol{r}_2 \times \dot{\boldsymbol{r}}_2 = (m_1 + m_2)\boldsymbol{R} \times \dot{\boldsymbol{R}} + \frac{m_1 m_2}{m_1 + m_2}\boldsymbol{r} \times \dot{\boldsymbol{r}} = \boldsymbol{L}_R + \boldsymbol{L}_r,$$

$$\boldsymbol{L}_R = m_1 \boldsymbol{R} \times \dot{\boldsymbol{r}}_1 + m_2 \boldsymbol{R} \times \dot{\boldsymbol{r}}_2 = (m_1 + m_2)\boldsymbol{R} \times \dot{\boldsymbol{R}},$$

$$\boldsymbol{L}_r = m_1(\boldsymbol{r}_1 - \boldsymbol{R}) \times \dot{\boldsymbol{r}}_1 + m_2(\boldsymbol{r}_2 - \boldsymbol{R}) \times \dot{\boldsymbol{r}}_2 = \frac{m_1 m_2}{m_1 + m_2}\boldsymbol{r} \times \dot{\boldsymbol{r}},$$

$$\boldsymbol{L}_r = m_1(\boldsymbol{r}_1 - \boldsymbol{R}) \times \left(\dot{\boldsymbol{r}}_1 - \dot{\boldsymbol{R}}\right) + m_2(\boldsymbol{r}_2 - \boldsymbol{R}) \times \left(\dot{\boldsymbol{r}}_2 - \dot{\boldsymbol{R}}\right).$$

The total angular momentum $\boldsymbol{L}$ is conserved whenever $\sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i = 0$ and the center of mass momentum $\boldsymbol{L}_R$ is conserved whenever $\sum_i \boldsymbol{F}_i = 0$. In both cases the choice of origin of coordinates is arbitrary and Galilean transformations are allowed.

Now let's consider the angular momentum relative to some arbitrary point $\boldsymbol{Q}(t)$,

$$\boldsymbol{L}_r' = \sum_i m_i(\boldsymbol{r}_i - \boldsymbol{Q}) \times \left(\dot{\boldsymbol{r}}_i - \dot{\boldsymbol{Q}}\right),$$

and answer the question when this angular momentum is conserved. Denote

$$M = \sum_i m_i \qquad \text{and} \qquad \boldsymbol{R} = \frac{1}{M}\sum_i m_i \boldsymbol{r}_i,$$

$$\boldsymbol{L} = \sum_i m_i \boldsymbol{r}_i \times \dot{\boldsymbol{r}}_i.$$

Then we can rewrite $\boldsymbol{L}_r'$ as follows by expanding the product of two parenthesis

$$\boldsymbol{L}_r' = \boldsymbol{L} - M\boldsymbol{R} \times \dot{\boldsymbol{Q}} - M\boldsymbol{Q} \times \dot{\boldsymbol{R}} + M\boldsymbol{Q} \times \dot{\boldsymbol{Q}},$$

and combine there a new product of two parenthesis

$$L_r' = L - MR \times \dot{R} + M(R - Q) \times \left(\dot{R} - \dot{Q}\right).$$

Note that $L_r = L - MR \times \dot{R}$, and therefore,

$$L_r' = L_r + M(R - Q) \times \left(\dot{R} - \dot{Q}\right),$$

$$\frac{dL_r'}{dt} = \frac{dL_r}{dt} + M(R - Q) \times \left(\ddot{R} - \ddot{Q}\right).$$

Therefore, we must have $\ddot{Q} = \ddot{R}$ in order for the quantity L_r' to be conserved, provided L_r is conserved. This show that $Q(t)$ should not differ from $R(t)$ other than by a Galilean transformation

$$Q(t) = R(t) + v_Q t + r_Q,$$

where v_Q and r_Q are arbitrary.

(c) For two particles interacting via [2]

$$V(r_1, r_2) = V_{12}(|r_1 - r_2|) + V_{ext}(|r_1|) + V_{ext}(|r_2|)$$

verify the conservation of total energy, total momentum, total angular momentum relative to the origin, and total angular momentum relative to the center of mass.
Lernziel: Learn to prove basics conservation laws for interacting particles.
Lösung: The total energy is conserved because the potential $V(r_1, r_2)$ does not explicitly depend on time. The same derivation as in 7(a) applies.

The total momentum is not conserved, because the potential $V(r_1, r_2)$ is not invariant under $r_i \to r_i + a$.

The total angular momentum is conserved only with respect to the origin $r_i = 0$. This is so because the potential $V(r_1, r_2)$ is invariant with respect to rotations around the origin: $r_i \to R r_i$, where R is a rotation matrix by an arbitrary angle round the point $r_i = 0$. For our example one can verify that $\sum_i r_i \times F_i = 0$, where the terms originating from V_{ext} give rise to radial forces and vanish when crossed with r_i, and the terms originating from V_{12} vanish when summed up, as we have seen in 7(a).

The total angular momentum with respect to the center of mass is not conserved, because the sum of all forces is not zero. In other words the center of mass does not

coincide with the origin. The origin is the only point of isotropy of the potential.