4 Hausaufgabe zu Theorie der höheren Mechanik zum Montag, den 17.5.2010

H7

a)

$$\vec{R_{1,z}} = \frac{\varrho \int\limits_0^h \int\limits_0^{2\pi} \int\limits_0^{a/2} \varrho z \; d\varrho d\varphi dz}{\varrho \int\limits_0^h \int\limits_0^{2\pi} \int\limits_0^{a/2} \varrho \; d\varrho d\varphi dz} = \frac{\frac{1}{8}a^2h^2\pi\varrho}{\frac{1}{4}a^2h\pi\varrho} = \frac{h}{2}$$

$$\vec{R_{2,z}} = \frac{\varrho \int\limits_h^{2h} \int\limits_0^{2\pi} \int\limits_0^{b/2} \varrho z \; d\varrho d\varphi dz}{\varrho \int\limits_h^{2h} \int\limits_0^{2\pi} \int\limits_0^{b/2} \varrho \; d\varrho d\varphi dz} = \frac{\frac{3}{8}b^2h^2\pi\varrho}{\frac{1}{4}b^2h\pi\varrho} = \frac{3h}{2}$$

$$\vec{R_{3,z}} = \frac{\varrho \int\limits_{2h}^{2h+d} \int\limits_0^{2\pi} \int\limits_0^{\sqrt{(d/2)^2-(z-2h-d/2)^2}} \varrho z \; d\varrho d\varphi dz}{\varrho \int\limits_{2h}^{2h+d} \int\limits_0^{2\pi} \int\limits_0^{\sqrt{(d/2)^2-(z-2h-d/2)^2}} \varrho \; d\varrho d\varphi dz} = \frac{\frac{1}{12}d^3(d+4h)\pi\varrho}{\frac{d^3\pi}{6}\varrho} = \frac{1}{2}d + 2h$$

$$\vec{R_z} = \frac{R_1M_1+R_2M_2+R_3M_3}{M_1+M_2+M_3} = \frac{\frac{1}{8}a^2h^2\pi\varrho+\frac{3}{8}b^2h^2\pi\varrho+\frac{1}{12}d^3(d+4h)\pi\varrho}{\frac{1}{4}a^2h\pi\varrho+\frac{1}{4}b^2h\pi\varrho+\frac{d^3\pi}{6}\varrho}$$

$$= \frac{3h^2\left(a^2+3b^2\right)+2d^4+8d^3h}{6h(a^2+b^2)+4d^3} \approx 0.925m$$

$$\Rightarrow R = (0, 0, 0.925m)$$

b)

$$\phi_1 = -G\varrho \int\limits_0^h \int\limits_0^{2\pi} \int\limits_0^{a/2} \frac{\varrho}{|\vec{r}|} \; d\varrho d\varphi dz = -G\varrho \int\limits_0^h \int\limits_0^{2\pi} \int\limits_0^{a/2} \frac{\varrho}{\sqrt{\varrho^2+z^2}} \; d\varrho d\varphi dz$$

$$= -2\pi G\varrho \int\limits_0^h \int\limits_0^{a/2} \frac{\varrho}{\sqrt{\varrho^2+z^2}} \; d\varrho dz = -2\pi G\varrho \int\limits_0^h \sqrt{(\frac{a}{2})^2+z^2} - z \; dz$$

$$= -2\pi G\varrho \left[\frac{1}{2}\left(z\sqrt{z^2+(\frac{a}{2})^2} + (\frac{a}{2})^2 \log\left(z+\sqrt{z^2+(\frac{a}{2})^2}\right)\right) - \frac{1}{2}z^2\right]_0^h$$

$$= -G\varrho\pi \left(\frac{1}{2}h\sqrt{a^2+4h^2} + \frac{1}{4}a^2\log\left(\sqrt{a^2+4h^2}+2h\right) - \frac{1}{8}a^2\log\left(a^2\right) - h^2\right)$$

$$\approx -0.255455 G\varrho$$

$$\phi_2 = -2\pi G\varrho \int\limits_h^{2h} \sqrt{(\frac{b}{2})^2+z^2} - z \; dz$$

$$= -2\pi G\varrho \left[\frac{1}{2}\left(z\sqrt{z^2+(\frac{b}{2})^2} + (\frac{b}{2})^2 \log\left(z+\sqrt{z^2+(\frac{b}{2})^2}\right)\right) - \frac{1}{2}z^2\right]_h^{2h}$$

$$= -2\pi G\varrho \frac{1}{8}\left(b^2\left(\log\left(\sqrt{b^2+16h^2}+4h\right)\right.\right.$$

$$\left.\left. - \log\left(\sqrt{b^2+4h^2}+2h\right)\right) - 2h\left(\sqrt{b^2+4h^2} - 2\sqrt{b^2+16h^2}+6h\right)\right)$$

$$\approx -0.134122 G\varrho$$

$$\phi_3 = -G\varrho\frac{\frac{4}{3}(\frac{d}{2})^3\pi}{\frac{1}{2}d+2h} = -G\varrho\frac{\frac{d^3\pi}{6}}{\frac{1}{2}d+2h} = -\frac{\pi d^3 G\varrho}{3(d+4h)} \approx -0.0538342G\varrho$$

$$\phi = \phi_1 + \phi_2 + \phi_3 = -0.443411G\varrho$$

H8

$\phi_{innen} = -4\pi G\varrho \int\limits_{r_1}^{r_2} r' \; dr' = 2\pi G\varrho(r_1^2 - r_2^2)$

$\phi_{aussen} = -4\pi G\varrho \int\limits_{r_1}^{r_2} \frac{r'^2}{r} \; dr' = \frac{4}{3}\pi G\varrho(r_1^3 - r_2^3)\frac{1}{r}$