4.14

The kinetic energy in terms of the Euler angles is

$$
\begin{aligned}
T &= \tfrac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) \\
&= \tfrac{1}{2}I_1(\dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi)^2 + \tfrac{1}{2}I_2(\dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi)^2 + \tfrac{1}{2}I_3(\dot\phi \cos\theta + \dot\psi)^2
\end{aligned}
$$

Then

$$
\begin{aligned}
\partial T/\partial\psi &= I_1(\dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi)(\dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi) \\
&\quad + I_2(\dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi)(-\dot\phi \sin\theta \sin\psi - \dot\theta \cos\psi) \\
&= I_1\omega_1\omega_2 + I_2(\omega_2)(-\omega_1) = (I_1 - I_2)\omega_1\omega_2
\end{aligned}
$$

$$\partial T/\partial\dot\psi = I_3(\dot\phi \cos\theta + \dot\psi) = I_3\omega_3$$

Then Lagrange's equation corresponding to ψ is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\psi}\right) - \frac{\partial T}{\partial\psi} = \Phi_\psi$$

or

$$I_3\dot\omega_3 + (I_2 - I_1)\omega_1\omega_2 = \Phi_\psi \tag{1}$$

This is Euler's third equation. The quantity Φ_ψ represents the generalized force corresponding to a rotation ψ about an axis and physically represents the component Λ_3 of the torque about this axis

The remaining equations

$$I_1\dot\omega_1 + (I_3 - I_2)\omega_2\omega_3 = \Lambda_1 \tag{2}$$

$$I_2\dot\omega_2 + (I_1 - I_3)\omega_3\omega_1 = \Lambda_2 \tag{3}$$

can be obtained from symmetry considerations by permutation of the indices. They are not directly obtained by using the Lagrange equations corresponding to θ and ϕ but can indirectly be deduced from them

4.15

(a) Kinetic energy $= T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$

$$= \tfrac{1}{2}ma^2\{[(1-\cos\theta)\dot{\theta}]^2 + [-\sin\theta\,\dot{\theta}]^2\}$$

$$= ma^2(1-\cos\theta)\dot{\theta}^2$$

Potential energy $= V = mgy = mga(1+\cos\theta)$

Then

Lagrangian $= L = T - V = ma^2(1-\cos\theta)\dot{\theta}^2 - mga(1+\cos\theta)$

(b) $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$, i.e. $\frac{d}{dt}[2ma^2(1-\cos\theta)\dot{\theta}] - [ma^2\sin\theta\,\dot{\theta}^2 + mga\sin\theta] = 0$

or
$$\frac{d}{dt}[(1-\cos\theta)\dot{\theta}] - \tfrac{1}{2}\sin\theta\,\dot{\theta}^2 - \frac{g}{2a}\sin\theta = 0$$

which can be written
$$(1-\cos\theta)\ddot{\theta} + \tfrac{1}{2}\sin\theta\,\dot{\theta}^2 - \frac{g}{2a}\sin\theta = 0$$

(c) If $u = \cos(\theta/2)$, then
$$\frac{du}{dt} = -\tfrac{1}{2}\sin(\theta/2)\dot{\theta}, \qquad \frac{d^2u}{dt^2} = -\tfrac{1}{2}\sin(\theta/2)\ddot{\theta} - \tfrac{1}{4}\cos(\theta/2)\dot{\theta}^2$$

Thus $\frac{d^2u}{dt^2} + \frac{g}{4a}u = 0$ is the same as
$$-\tfrac{1}{2}\sin(\theta/2)\ddot{\theta} - \tfrac{1}{4}\cos(\theta/2)\dot{\theta}^2 + \frac{g}{4a}\cos(\theta/2) = 0$$

which can be written as
$$\ddot{\theta} + \tfrac{1}{2}\cot(\theta/2)\dot{\theta}^2 - \frac{g}{2a}\cot(\theta/2) = 0 \qquad (1)$$

Since
$$\cot(\theta/2) = \frac{\cos(\theta/2)}{\sin(\theta/2)} = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)} = \frac{\sin\theta}{1-\cos\theta}$$

it follows that equation (1) is the same as that obtained in (b)

(d) The solution of the equation is
$$u = \cos(\theta/2) = c_1\cos\sqrt{4a/g}\,t + c_2\sin\sqrt{4a/g}\,t$$

from which we see that $\cos(\theta/2)$ returns to its original value after a time $2\pi\sqrt{4a/g}$ which is the required period. Note that this period is the same as that of a simple pendulum with length $l = 4a$.