#18

a) Let $\mathbf{A} = \boldsymbol{\omega}$ in Problem P18 Then

$$\left.\frac{d\boldsymbol{\omega}}{dt}\right|_F = \left.\frac{d\boldsymbol{\omega}}{dt}\right|_M + \boldsymbol{\omega} \times \boldsymbol{\omega} = \left.\frac{d\boldsymbol{\omega}}{dt}\right|_M$$

Since $d\boldsymbol{\omega}/dt$ is the angular acceleration, the required statement is proved.

b)

Replacing $\mathbf{A}$ by the position vector $\mathbf{r}$ of the particle, we have

$$\left.\frac{d\mathbf{r}}{dt}\right|_F = \left.\frac{d\mathbf{r}}{dt}\right|_M + \boldsymbol{\omega} \times \mathbf{r} \tag{1}$$

If $\mathbf{r}$ is expressed in terms of the unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ of the moving coordinate system, then the velocity of the particle relative to this system is, on dropping the subscript M,

$$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} \tag{2}$$

and the velocity of the particle relative to the fixed system is from (1)

$$\left.\frac{d\mathbf{r}}{dt}\right|_F = \frac{d\mathbf{r}}{dt} + \boldsymbol{\omega} \times \mathbf{r} \tag{3}$$

The velocity (3) is sometimes called the *true velocity*, while (2) is the *apparent velocity*.

c)

$$D_F \equiv D_M + \boldsymbol{\omega} \times$$

By definition $\quad D_F\mathbf{A} = \left.\dfrac{d\mathbf{A}}{dt}\right|_F = $ derivative in fixed system

$D_M\mathbf{A} = \left.\dfrac{d\mathbf{A}}{dt}\right|_M = $ derivative in moving system

Then from Problem P18

$$D_F\mathbf{A} = D_M\mathbf{A} + \boldsymbol{\omega} \times \mathbf{A} = (D_M + \boldsymbol{\omega} \times)\mathbf{A}$$

which shows the equivalence of the operators $D_F \equiv D_M + \boldsymbol{\omega} \times$.

The acceleration of the particle as seen by the observer in the fixed XYZ system is $D_F^2\mathbf{r} = D_F(D_F\mathbf{r})$. Using the operator equivalence we have

$$\begin{aligned}
D_F(D_F\mathbf{r}) &= D_F(D_M\mathbf{r} + \boldsymbol{\omega} \times \mathbf{r}) \\
&= (D_M + \boldsymbol{\omega} \times)(D_M\mathbf{r} + \boldsymbol{\omega} \times \mathbf{r}) \\
&= D_M(D_M\mathbf{r} + \boldsymbol{\omega} \times \mathbf{r}) + \boldsymbol{\omega} \times (D_M\mathbf{r} + \boldsymbol{\omega} \times \mathbf{r}) \\
&= D_M^2\mathbf{r} + D_M(\boldsymbol{\omega} \times \mathbf{r}) + \boldsymbol{\omega} \times D_M\mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})
\end{aligned}$$

or since $D_M(\boldsymbol{\omega} \times \mathbf{r}) = (D_M\boldsymbol{\omega}) \times \mathbf{r} + \boldsymbol{\omega} \times (D_M\mathbf{r})$,

$$D_F^2\mathbf{r} = D_M^2\mathbf{r} + (D_M\boldsymbol{\omega}) \times \mathbf{r} + 2\boldsymbol{\omega} \times (D_M\mathbf{r}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \tag{1}$$

If $\mathbf{r}$ is the position vector expressed in terms of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ of the moving coordinate system, then the acceleration of the particle relative to this system is, on dropping the subscript M,

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k} \tag{2}$$

The acceleration of the particle relative to the fixed system is given from (1) as

$$\left.\frac{d^2\mathbf{r}}{dt^2}\right|_F = \frac{d^2\mathbf{r}}{dt^2} + \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} + 2\boldsymbol{\omega} \times \left(\frac{d\mathbf{r}}{dt}\right) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \tag{3}$$

The acceleration (3) is sometimes called the *true acceleration*, while (2) is the *apparent acceleration*.

H19

(a) Inertial system B

$$\boxed{\begin{array}{lll} \ddot{x}_1 = 0 , & x_1(0) = 0 , & \dot{x}_1(0) = v_1 \\ \ddot{x}_2 = 0 , & x_2(0) = 0 , & \dot{x}_2(0) = v_2 \\ \ddot{x}_3 = -g , & x_3(0) = 0 , & \dot{x}_3(0) = v_3 \end{array}} \xrightarrow{\text{Integration}} \begin{array}{l} x_1(t) = v_1 t \\ x_2(t) = v_2 t \\ x_3(t) = v_3 t - \frac{g}{2} t^2 \end{array}$$

$$\left.\begin{array}{l} x_1' = \cos\omega t \, x_1 + \sin\omega t \, x_2 \\ x_2' = -\sin\omega t \, x_1 + \cos\omega t \, x_2 \\ x_3' = x_3 \end{array}\right\} \rightarrow \boxed{\begin{array}{l} x_1'(t) = v_1 t \cos\omega t + v_2 t \sin\omega t \\ x_2'(t) = -v_1 t \sin\omega t + v_2 t \cos\omega t \\ x_3'(t) = v_3 t - \frac{g}{2} t^2 \end{array}}$$

(b) Kartesisch

$$\boxed{\begin{array}{ll} \ddot{x}_1' = 2\omega \dot{x}_2' + \omega^2 x_1' & x_1'(0) = 0 , \quad \dot{x}_1'(0) = v_1 \\ \ddot{x}_2' = -2\omega \dot{x}_1' + \omega^2 x_2' & x_2'(0) = 0 , \quad \dot{x}_2'(0) = v_2 \end{array}} \quad \begin{array}{l} (B.b.1) \\ (B.b.2) \end{array}$$

$$u := x_1' + i x_2' \xrightarrow{(B.b.1) + i (B.b.2)} \boxed{\ddot{u} + 2i\omega \dot{u} - \omega^2 u = 0}$$

Ansatz: $u \sim e^{\lambda t}$ liefert charakteristisches Polynom

$$\lambda^2 + 2i\omega\lambda - \omega^2 = (\lambda + i\omega)^2 = 0$$

$$\rightarrow \lambda_{1/2} = -i\omega \quad \text{(Doppelwurzel)}$$

$$\rightarrow u(t) = (A + Bt) e^{-i\omega t}$$

Anfangsbed. liefern $A = 0$, $B = v_1 + i v_2$

$$\boxed{\begin{array}{l} x_1'(t) = v_1 t \cos\omega t + v_2 t \sin\omega t = v t \cos(\varphi_0 - \omega t) \\ x_2'(t) = -v_1 t \sin\omega t + v_2 t \cos\omega t = v t \sin(\varphi_0 - \omega t) \end{array}} \quad \tan\varphi_0 = \frac{v_2}{v_1}$$