H13] geänderte Notation:
$\omega_1 = p \quad \omega_2 = q \quad \omega_3 = r$

a) Trägheitsmoment um die Hantelachse: $\Theta_A = 2\Theta_{\text{Kugel}} = \frac{4}{5}mr^2$. Trägheitsmomente um zwei Achsen in der Ebene senkrecht dazu: $\Theta_S = \Theta_A + \frac{1}{2}ma^2$ (Steiner). Man findet also $\Theta_3 = \Theta_A < \Theta_S = \Theta_1 = \Theta_2$, d.h. es handelt sich um einen symmetrischen Kreisel.

b) Die kräftefreien Eulerschen Kreiselgleichungen lauten $\Theta_1\dot\omega_1 - (\Theta_2 - \Theta_3)\omega_2\omega_3 = 0$ und zyklische Permutationen. Man findet also

$$\dot\omega_1 - \gamma\omega_2\omega_3 = 0, \qquad \dot\omega_2 + \gamma\omega_3\omega_1 = 0, \qquad \dot\omega_3 = 0, \tag{5}$$

mit

$$\gamma = 1 - \frac{\Theta_A}{\Theta_S} = \left(1 + \frac{8}{5}\frac{r^2}{a^2}\right)^{-1}.$$

Stabilitätsanalyse: Es sei der momentane Drehvektor $\vec\omega$ nahe der Hantelachse $\hat e_3$, d.h. $\omega_3 = \omega_0 + \delta_3$, $\omega_1 = \delta_1$, $\omega_2 = \delta_2$ mit kleinen δ_i. Die linearisierten Kreiselgleichungen für die Komponenten senkrecht zur Hantelachse lauten also:

$$\begin{aligned} \dot\delta_1 - \gamma\omega_0\delta_2 &= 0 \\ \dot\delta_2 + \gamma\omega_0\delta_1 &= 0 \end{aligned}$$

Die Zeitableitung der einen eingesetzt in die jeweils andere Gleichung ergibt

$$\ddot\delta_i + \gamma^2\omega_0^2\delta_i = 0, \qquad i = 1,2$$

und beschreibt somit eine stabile Schwingung mit $\nu = \gamma\omega_0$.

c) Die allgemeine Lösung von (5) ist $\omega_3 = cst = \omega_0\cos\alpha$, und

$$\omega_1 = \omega_\perp\cos(\gamma\omega_3 t + \phi) \tag{6}$$
$$\omega_2 = \omega_\perp\sin(\gamma\omega_3 t + \phi) \tag{7}$$

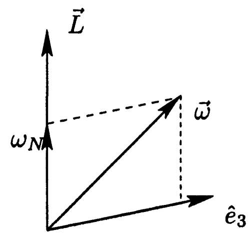
mit $\omega_0^2 = \omega_3^2 + \omega_\perp^2$, d.h. $\omega_\perp = \omega_0\sin\alpha$.

Dann gilt im körperfesten System

$$\vec L = \underline{\Theta}\vec\omega = \Theta_S\vec\omega_\perp + \Theta_A\omega_3\hat e_3 = \Theta_S(\omega - \gamma\omega_3\hat e_3) \tag{8}$$

nach Definition von γ.

d) In der Zerlegung (8) ist der Koeffizient von $\vec L$ in Richtung der Hantelachse $\hat e_3$ negativ, da $\gamma > 0$, d.h. im Laborsystem liegt der momentane Drehvektor zwischen dem raumfesten Drehimpulsvektor und der Hantelachse (s. Skizze). Hantelachse und momentaner Drehvektor beschreiben also einen Kegel mit der Frequenz ω_N um den Drehimpulsvektor. Aus (8) folgt



$$\vec\omega = \frac{1}{\Theta_S}\vec L + \gamma\omega_3\hat e_3 .$$

Nun ist jedoch die Nutationsfrequenz gleich der Projektion von $\vec\omega$ auf $\vec L$ entlang $\hat e_3$ und somit

$$\begin{aligned} \omega_N &= \frac{|\vec L|}{\Theta_S} = \frac{1}{\Theta_S}\sqrt{\Theta_S^2\omega_\perp^2 + \Theta_A^2\omega_3^2} \\ &= \omega_0\sqrt{1 + ((1-\gamma)^2 - 1)\cos^2\alpha} \\ &= \omega_0\sqrt{1 + \gamma(\gamma - 2)\cos^2\alpha} \end{aligned}$$

Bei kleiner Auslenkung aus der Hantelachse ist $\omega_3 \approx \omega_0$ bzw. $\cos\alpha \approx 1$, und man findet

$$\omega_N = \omega_0\sqrt{1 + \gamma(\gamma - 2)} = \omega_0(1 - \gamma) = \nu - \omega_0$$

Im Klartext: die Nutationsfrequenz im Laborsystem ist einfach die Frequenz im körperfesten System minus die Roationsfrequenz des Körpers selbst.

H 14

The kinetic energy in terms of the Euler angles is

$$
\begin{aligned}
T &= \tfrac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) \\
&= \tfrac{1}{2}I_1(\dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi)^2 + \tfrac{1}{2}I_2(\dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi)^2 + \tfrac{1}{2}I_3(\dot\phi \cos\theta + \dot\psi)^2
\end{aligned}
$$

Then

$$
\begin{aligned}
\partial T/\partial\psi &= I_1(\dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi)(\dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi) \\
&\quad + I_2(\dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi)(-\dot\phi \sin\theta \sin\psi - \dot\theta \cos\psi) \\
&= I_1\omega_1\omega_2 + I_2(\omega_2)(-\omega_1) = (I_1 - I_2)\omega_1\omega_2
\end{aligned}
$$

$$
\partial T/\partial\dot\psi = I_3(\dot\phi \cos\theta + \dot\psi) = I_3\omega_3
$$

Then Lagrange's equation corresponding to ψ is

$$
\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\psi}\right) - \frac{\partial T}{\partial\psi} = \Phi_\psi
$$

or

$$
I_3\dot\omega_3 + (I_2 - I_1)\omega_1\omega_2 = \Phi_\psi \qquad (1)
$$

This is Euler's third equation The quantity Φ_ψ represents the generalized force corresponding to a rotation ψ about an axis and physically represents the component Λ_3 of the torque about this axis

The remaining equations

$$
I_1\dot\omega_1 + (I_3 - I_2)\omega_2\omega_3 = \Lambda_1 \qquad (2)
$$

$$
I_2\dot\omega_2 + (I_1 - I_3)\omega_3\omega_1 = \Lambda_2 \qquad (3)
$$

can be obtained from symmetry considerations by permutation of the indices. They are not directly obtained by using the Lagrange equations corresponding to θ and ϕ but can indirectly be deduced from them

415

(a) Kinetic energy $= T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$

$$= \tfrac{1}{2}ma^2\{[(1-\cos\theta)\dot{\theta}]^2 + [-\sin\theta\,\dot{\theta}]^2\}$$

$$= ma^2(1-\cos\theta)\dot{\theta}^2$$

Potential energy $= V = mgy = mga(1+\cos\theta)$

Then

Lagrangian $= L = T - V = ma^2(1-\cos\theta)\dot{\theta}^2 - mga(1+\cos\theta)$

(b) $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial\dot{\theta}}\right) - \dfrac{\partial L}{\partial\theta} = 0$, i.e. $\dfrac{d}{dt}[2ma^2(1-\cos\theta)\dot{\theta}] - [ma^2\sin\theta\,\dot{\theta}^2 + mga\sin\theta] = 0$

or
$$\frac{d}{dt}[(1-\cos\theta)\dot{\theta}] - \tfrac{1}{2}\sin\theta\,\dot{\theta}^2 - \frac{g}{2a}\sin\theta = 0$$

which can be written
$$(1-\cos\theta)\ddot{\theta} + \tfrac{1}{2}\sin\theta\,\dot{\theta}^2 - \frac{g}{2a}\sin\theta = 0$$

(c) If $u = \cos(\theta/2)$, then

$$\frac{du}{dt} = -\tfrac{1}{2}\sin(\theta/2)\dot{\theta}, \qquad \frac{d^2u}{dt^2} = -\tfrac{1}{2}\sin(\theta/2)\ddot{\theta} - \tfrac{1}{4}\cos(\theta/2)\dot{\theta}^2$$

Thus $\dfrac{d^2u}{dt^2} + \dfrac{g}{4a}u = 0$ is the same as

$$-\tfrac{1}{2}\sin(\theta/2)\ddot{\theta} - \tfrac{1}{4}\cos(\theta/2)\dot{\theta}^2 + \frac{g}{4a}\cos(\theta/2) = 0$$

which can be written as

$$\ddot{\theta} + \tfrac{1}{2}\cot(\theta/2)\dot{\theta}^2 - \frac{g}{2a}\cot(\theta/2) = 0 \qquad (1)$$

Since
$$\cot(\theta/2) = \frac{\cos(\theta/2)}{\sin(\theta/2)} = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)} = \frac{\sin\theta}{1-\cos\theta}$$

it follows that equation (1) is the same as that obtained in (b)

(d) The solution of the equation is

$$u = \cos(\theta/2) = c_1\cos\sqrt{4a/g}\,t + c_2\sin\sqrt{4a/g}\,t$$

from which we see that $\cos(\theta/2)$ returns to its original value after a time $2\pi\sqrt{4a/g}$ which is the required period. Note that this period is the same as that of a simple pendulum with length $l = 4a$.

P14

(a) Let the ellipse be chosen in the xy plane of Fig. 14-1. The particle of mass m moving on the ellipse has coordinates (x, y). However, since we have the transformation equations $x = a \cos \theta$, $y = b \sin \theta$, we can specify the motion completely by use of the generalized coordinate θ.

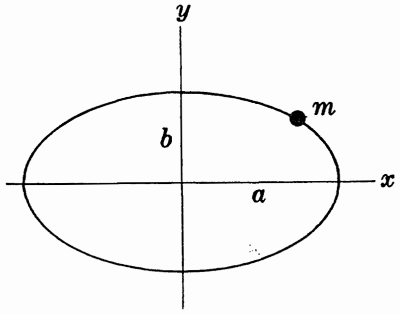


Fig.41-1

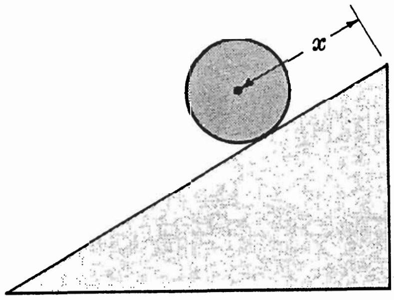


Fig.41-2

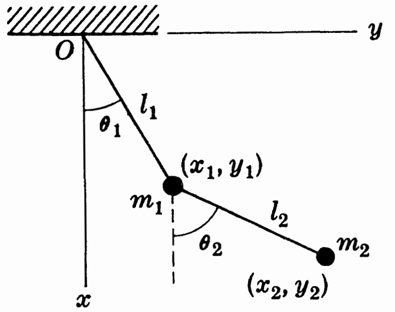


Fig.41-3

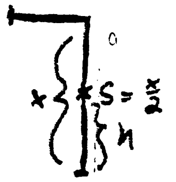
(b) The position of the cylinder [Fig. 14-2 above] on the inclined plane can be completely specified by giving the distance x traveled by the center of mass and the angle θ of rotation turned through by the cylinder about its axis.

If there is no slipping, x is related to θ so that only one generalized coordinate [either x or θ] is needed. If there is slipping, two generalized coordinates x and θ are needed.

(c) Two coordinates θ_1 and θ_2 completely specify the positions of masses m_1 and m_2 [see Fig. 14-3 above] and can be considered as the required generalized coordinates.

Gesamtlänge l, Massendichte ρ

$\boxed{H16}$



kinetische Energie: $T = \frac{1}{2} m_g \dot{x}^2 = \frac{1}{2} \cdot \rho l \cdot \dot{x}^2$

potentielle Energie: $V = m_x g h = g \cdot \rho x \cdot \frac{x}{2} = -\frac{1}{2} g \rho x^2$

a) Lagrangefunktion: $L = T - V = \frac{1}{2} \rho l \dot{x}^2 + \frac{1}{2} g \rho x^2 = \frac{1}{2} \rho \left(l \dot{x}^2 + g x^2 \right)$

b) Bewegungsgleichung: $\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$

$$\Leftrightarrow \frac{d}{dt} (\rho l \dot{x}) - g \rho x = 0$$

$$\Leftrightarrow \boxed{\rho l \ddot{x} - g \rho x = 0} \quad \Rightarrow \quad \ddot{x} - \frac{g}{l} x = 0$$

c) Lösen:

Lösungsansatz: $x = e^{+\lambda t}$ $\quad \dot{x} = +\lambda e^{+\lambda t}$ $\quad \ddot{x} = \lambda^2 e^{+\lambda t}$

$$\lambda^2 e^{+\lambda t} - \frac{g}{l} e^{+\lambda t} = 0 \Leftrightarrow \lambda^2 - \frac{g}{l} = 0 \Rightarrow \lambda = \pm \sqrt{\frac{g}{l}}$$

allg. Lsg: $x = A e^{\sqrt{\frac{g}{l}} t} + B e^{-\sqrt{g/l}\, t}$, $\quad \dot{x} = A \sqrt{\frac{g}{l}} e^{\sqrt{} t} - B \sqrt{\frac{g}{l}} e^{-\sqrt{} t}$

Anfangsbed. $x(0) = a$, $\dot{x}(0) = 0$

$$x(0) = A e^0 + B e^0 = A + B \stackrel{!}{=} a$$

$$\dot{x}(0) = A \sqrt{\frac{g}{l}} - B \sqrt{\frac{g}{l}} \stackrel{!}{=} 0 \Rightarrow \sqrt{\frac{g}{l}} (A - B) = 0 \Rightarrow A - B = 0$$

also: $\left| \begin{matrix} A + B = a \\ A - B = 0 \end{matrix} \right| \Rightarrow A = \frac{a}{2}, \; B = \frac{a}{2}$

allg. Lsg.: $x = \frac{a}{2} e^{\sqrt{\frac{g}{l}} t} + \frac{a}{2} e^{-\sqrt{g/l}\, t}$

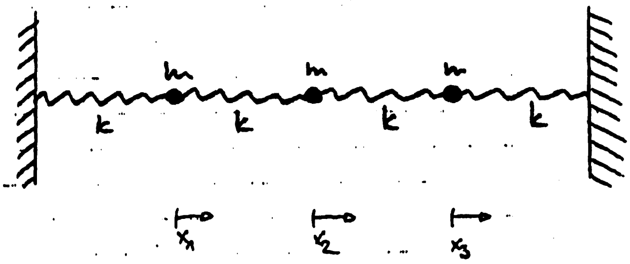
$= \frac{a}{2} \left(e^{\sqrt{} t} + e^{-\sqrt{} t} \right)$ $\qquad \cosh x = \frac{e^x + e^{-x}}{2}$

$$\boxed{x = a \cosh\left(\sqrt{\frac{g}{l}}\, t \right)}$$

H17

3 Massen in einer Reihe (nur longitudinale Schwingung)

(a) → Bewegungsgleichungen



$$\ddot{x}_1 + \frac{k}{m}(2x_1 - x_2) = 0$$
$$\ddot{x}_2 + \frac{k}{m}(2x_2 - x_1 - x_3) = 0$$
$$\ddot{x}_3 + \frac{k}{m}(2x_3 - x_2) = 0$$

(b) Ansatz $x_i(t) = a_i \cos(\omega t - \delta)$ $\Longrightarrow$

$$\begin{pmatrix} 2\frac{k}{m} - \omega^2 & -\frac{k}{m} & 0 \\ -\frac{k}{m} & 2\frac{k}{m} - \omega^2 & -\frac{k}{m} \\ 0 & -\frac{k}{m} & 2\frac{k}{m} - \omega^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad (1)$$

Determinante der Matrix muß 0 sein, andernfalls existiert die inverse Matrix und es folgt mit ($M\vec{a} = \vec{0} \Rightarrow M^{-1}M\vec{a} = M^{-1}\vec{0} \Rightarrow \vec{a} = \vec{0}$), nur die triviale Lösung $\vec{a} = \vec{0}$.

$$\Rightarrow \left[\left(2\frac{k}{m} - \omega^2\right)^3 - 2\left(\frac{k}{m}\right)^2 \left(2\frac{k}{m} - \omega^2\right) \right] = 0$$

$$\Rightarrow \left(2\frac{k}{m} - \omega^2\right) \left[\omega^4 - 4\frac{k\omega^2}{m} + 4\left(\frac{k}{m}\right)^2 - 2\left(\frac{k}{m}\right)^2 \right] = 0$$

$$\Rightarrow \omega_1^2 = 2\frac{k}{m}$$

$$\Rightarrow \left[\omega^4 - 4\frac{k}{m}\omega^2 + 2\left(\frac{k}{m}\right)^2 \right] = 0$$

$$\Rightarrow (\omega^2)_{2/3} = 2\frac{k}{m} \pm \sqrt{4\left(\frac{k}{m}\right)^2 - 2\left(\frac{k}{m}\right)^2}$$

$$= \frac{k}{m}\left(2 \pm \sqrt{2}\right)$$

$$\Rightarrow \boxed{\omega_1^2 = 2\frac{k}{m}, \quad \omega_2^2 = (2+\sqrt{2})\frac{k}{m}, \quad \omega_3^2 = (2-\sqrt{2})\frac{k}{m}}$$

(c) Berechnung der zugehörigen Eigenvektoren

$\omega_1^2 = 2\frac{k}{m}$

a_1	a_2	a_3	
$2\frac{k}{m} - 2\frac{k}{m}$	$-\frac{k}{m}$	0	0
$-\frac{k}{m}$	$2\frac{k}{m} - 2\frac{k}{m}$	$-\frac{k}{m}$	0
0	$-\frac{k}{m}$	$2\frac{k}{m} - 2\frac{k}{m}$	0
0	1	0	0
1	0	1	0
0	1	0	0

$\Rightarrow \quad a_2 = 0, \quad a_1 = -a_3$

$\Rightarrow$ normierter Eigenvektor: $\vec{V}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

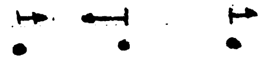
Schwingung

$\omega_2^2 = (2+\sqrt{2})\frac{k}{m}$

a_1	a_2	a_3		
$2\frac{k}{m} - (2+\sqrt{2})\frac{k}{m}$	$-\frac{k}{m}$	0	0	
$-\frac{k}{m}$	$2\frac{k}{m} - (2+\sqrt{2})\frac{k}{m}$	$-\frac{k}{m}$	0	
0	$-\frac{k}{m}$	$2\frac{k}{m} - (2+\sqrt{2})\frac{k}{m}$	0	
$\sqrt{2}$	1	0	0	$]-$
1	$\sqrt{2}$	1	0	$\vert \cdot \sqrt{2}\ ]-$
0	1	$\sqrt{2}$	0	
$\sqrt{2}$	1	0	0	
0	-1	$-\sqrt{2}$	0	$]+$
0	1	$\sqrt{2}$	0	$]+$
$\sqrt{2}$	1	0	0	
0	1	$\sqrt{2}$	0	
0	0	0	0	

$\Rightarrow \quad a_2 = a \quad \Rightarrow \quad a_1 = -\frac{1}{\sqrt{2}}\, a \,, \quad a_3 = -\frac{1}{\sqrt{2}}\, a$

Schwingung



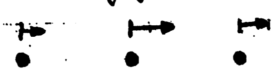
$\longrightarrow$ normierter Eigenvektor: $\vec{v}_2 = \begin{pmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{pmatrix}$

$\omega_3^2 = (2-\sqrt{2})\,\frac{k}{m}$

a_1	a_2	a_3		
$2\frac{k}{m}-(2-\sqrt{2})\frac{k}{m}$	$-\frac{k}{m}$	0	0	
$-\frac{k}{m}$	$2\frac{k}{m}-(2-\sqrt{2})\frac{k}{m}$	$-\frac{k}{m}$	0	
0	$-\frac{k}{m}$	$2\frac{k}{m}-(2-\sqrt{2})\frac{k}{m}$	0	
$\sqrt{2}$	-1	0	0	
-1	$\sqrt{2}$	-1	0	$\vert\cdot\sqrt{2}$]$_+$
0	-1	$\sqrt{2}$	0	
$\sqrt{2}$	-1	0	0	
0	1	$-\sqrt{2}$	0	
0	-1	$\sqrt{2}$	0	]$_+$
$\sqrt{2}$	-1	0	0	
0	1	$-\sqrt{2}$	0	
0	0	0	0	

$\Rightarrow \quad a_2 = a \quad \Rightarrow \quad a_3 = \frac{1}{\sqrt{2}}\, a \,, \quad a_1 = \frac{1}{\sqrt{2}}\, a$

Schwingung



$\longrightarrow$ normierter Eigenvektor: $\vec{v}_3 = \begin{pmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{pmatrix}$

(d) → Matrix aus diesen Eigenvektoren

$$\boxed{A = (\vec{v}_1, \vec{v}_2, \vec{v}_3) = \begin{pmatrix} 1/\sqrt{2} & 1/2 & 1/2 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/2 & 1/2 \end{pmatrix}}$$

$$A^T \cdot A = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/2 & -1/\sqrt{2} & 1/2 \\ 1/2 & 1/\sqrt{2} & 1/2 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/2 & 1/2 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/2 & 1/2 \end{pmatrix} = \mathbb{1}_{3\times3} \checkmark$$

also $A^T = A^{-1}$ ist orthogonal; ist auch sofort einleuchtend, da die Eigenvektoren für (nichtentartete) Eigenwerte immer orthogonal zueinander sind [und man außerdem diese Eigenvektoren noch normiert hat]. Somit ist Spaltenorthonormalität ($\Leftrightarrow$ Orthogonalmatrix) automatisch gegeben.

(e) Kinetische Energie der Anordnung

$$\underline{\underline{T}} = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 + \frac{1}{2}m\dot{x}_3^2 = \underline{\underline{\frac{1}{2}m\left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2\right)}}$$

Dies läßt sich umschreiben in

$$T = \frac{1}{2}(\dot{x}_1, \dot{x}_2, \dot{x}_3) \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \frac{1}{2}\dot{\vec{x}}^T \hat{T} \dot{\vec{x}}$$ (Matrixschreibweise)

$\Rightarrow$ Matrix der kinetischen Energie $\hat{T}$ ist diagonal: $\underline{\hat{T} = m \cdot \mathbb{1}_{3\times3}}$

potentielle Energie der Anordnung

$$\underline{\underline{V}} = \frac{1}{2}k\left(x_1^2 + (x_1-x_2)^2 + (x_2-x_3)^2 + x_3^2 \right)$$

$$= \underline{\underline{\frac{1}{2}k\left(2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3 \right)}}$$

Dies läßt sich umschreiben zu

$$\underline{\underline{V}} = \frac{1}{2}k(x_1, x_2, x_3) \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underline{\underline{\frac{1}{2}\vec{x}^T \hat{V} \vec{x}}}$$

$\Rightarrow$ die Matrix der potentiellen Energie ist nicht-diagonal

$$\hat{V} = k \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

(#) Mit diesen beiden Matrizen $\hat{T}$ & $\hat{V}$ läßt sich das Gleichungssystem ① auch darstellen als

$$(\hat{V} - \omega^2 \hat{T})\, \vec{a} = \vec{0} \quad , \text{ denn}$$

$$\begin{pmatrix} 2k - m\omega^2 & -k & 0 \\ -k & 2k - m\omega^2 & -k \\ 0 & -k & 2k - m\omega^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

[" Division durch die Masse m ergibt das Gleichungssystem ① "]

(g) Zu zeigen: Prozedur $A^T \hat{V} A$ diagonalisiert die Matrix der potentiellen Energie

[die kinetische Energie ist bereits diagonal, deswegen gilt trivialerweise

$$\hat{T}' = A^T \hat{T} A = m \cdot A^T \cdot \mathbb{1}_3 \cdot A = m\, A^T \cdot A = m\, \mathbb{1}_3 = \hat{T} \quad]$$

$$\hat{V}' = A^T \hat{V} A = k \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2+\sqrt{2} & 0 \\ 0 & 0 & 2-\sqrt{2} \end{pmatrix} = m \cdot \begin{pmatrix} \omega_1^2 & 0 & 0 \\ 0 & \omega_2^2 & 0 \\ 0 & 0 & \omega_3^2 \end{pmatrix}$$

$\Rightarrow$ man erhält eine Diagonalmatrix, deren Einträge gerade die Eigenfrequenzen sind (multipliziert mit m)

(h) Einführung der neuen Koordinaten $\vec{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$ mittels $\vec{X}^T = \vec{x}^T \cdot A$!

$$\Rightarrow \quad X_1 = \frac{1}{\sqrt{2}} x_1 \qquad - \frac{1}{\sqrt{2}} x_3$$
$$X_2 = \frac{1}{2} x_1 - \frac{1}{\sqrt{2}} x_2 + \frac{1}{2} x_3$$
$$X_3 = \frac{1}{2} x_1 + \frac{1}{\sqrt{2}} x_2 + \frac{1}{2} x_3$$

Bewegungsgleichungen für diese neuen Koordinaten:

$$\ddot{X}_1 = \frac{1}{\sqrt{2}} \ddot{x}_1 - \frac{1}{\sqrt{2}} \ddot{x}_3 = -\frac{k}{m}\left(\sqrt{2} x_1 - \frac{1}{\sqrt{2}} x_2\right) + \frac{k}{m}\left(\sqrt{2} x_3 - \frac{1}{\sqrt{2}} x_2\right)$$
$$= \frac{k}{m}\sqrt{2}\,(-x_1 + x_3) = -\frac{k}{m} \cdot 2 \cdot X_1$$

$$\Rightarrow \quad \ddot{X}_1 + 2\frac{k}{m} X_1 = 0 \quad \Rightarrow \quad \boxed{\ddot{X}_1 + \omega_1^2 X_1 = 0}$$

$$\ddot{X}_2 = \frac{1}{2}\ddot{x}_1 - \frac{1}{\sqrt{2}} \ddot{x}_2 + \frac{1}{2}\ddot{x}_3 = -\frac{k}{m}\left(x_1 - \frac{1}{2} x_2\right) + \frac{k}{m}\left(\sqrt{2} x_2 - \frac{1}{\sqrt{2}} x_1 - \frac{1}{\sqrt{2}} x_3\right) - \frac{k}{m}\left(x_3 - \frac{1}{2} x_2\right)$$
$$= -\frac{k}{m}\left(\left(1 + \frac{1}{\sqrt{2}}\right) x_1 - \left(1 + \sqrt{2}\right) x_2 + \left(1 + \frac{1}{\sqrt{2}}\right) x_3 \right)$$

$$\text{mit} \quad 1 + \frac{1}{\sqrt{2}} = \frac{2+\sqrt{2}}{2}, \quad (1+\sqrt{2}) = \frac{2+\sqrt{2}}{\sqrt{2}}$$

$$= -\frac{k}{m}\left((2+\sqrt{2}) \frac{x_1}{2} - (2+\sqrt{2}) \frac{x_2}{\sqrt{2}} + (2+\sqrt{2}) \frac{x_3}{2} \right)$$
$$= -\frac{k}{m}(2+\sqrt{2}) \quad X_2$$

$$\Rightarrow \quad \ddot{X}_2 + (2+\sqrt{2})\frac{k}{m} X_2 = 0 \quad \Rightarrow \quad \boxed{\ddot{X}_2 + \omega_2^2 X_2 = 0}$$

$$\ddot{X}_3 = \frac{1}{2}\ddot{x}_1 + \frac{1}{\sqrt{2}}\ddot{x}_2 + \frac{1}{2}\ddot{x}_3 = -\frac{k}{m}\left(x_1 - \frac{1}{2}x_2\right) - \frac{k}{m}\left(\sqrt{2}x_2 - \frac{1}{\sqrt{2}}x_1 - \frac{1}{\sqrt{2}}x_3\right) - \frac{k}{m}\left(x_3 - \frac{1}{2}x_2\right)$$

$$= -\frac{k}{m}\left(\left(1-\frac{1}{\sqrt{2}}\right)x_1 + \left(1-\sqrt{2}\right)x_2 + \left(1-\frac{1}{\sqrt{2}}\right)x_3\right)$$

mit $1-\frac{1}{\sqrt{2}} = \frac{2-\sqrt{2}}{2}$, $(1-\sqrt{2}) = \frac{2-\sqrt{2}}{\sqrt{2}}$

$$= -\frac{k}{m}\left((2-\sqrt{2})\frac{x_1}{2} + (2-\sqrt{2})\frac{x_2}{\sqrt{2}} + (2-\sqrt{2})\frac{x_3}{2}\right)$$

$$= -\frac{k}{m}(2-\sqrt{2}) X_3$$

$$\Rightarrow \quad \ddot{X}_3 + (2-\sqrt{2})\frac{k}{m} X_3 = 0 \quad \Rightarrow \quad \boxed{\ddot{X}_3 + \omega_3^2 X_3 = 0}$$

$\Rightarrow$ in den neuen Koordinaten sind die Differentialgleichungen entkoppelt! Die dabei auftretenden Lösungsfrequenzen sind gerade die Eigenfrequenzen des Systems

$$L = (q_1, q_2, \dot q_1, \dot q_2, t) \qquad \frac{d}{dt}\delta q = \delta \dot q$$

$\boxed{P15}$ $\delta q_i(t_1) = \delta q_i(t_2) = 0$

$$\delta S = S[q+\delta q] - S[q]$$

Hamilton'sches Prinzip: $\delta S = \delta \int_{t_1}^{t_2} L\, dt = 0$

$$\Rightarrow \delta S = \int_{t_1}^{t_2} \left(L(q_1+\delta q_1, q_2+\delta q_2, \dot q_1+\delta \dot q_1, \dot q_2+\delta \dot q_2, t) - L(q_1, q_2, \dot q_1, \dot q_2, t)\right) dt = 0$$

$$= \int_{t_1}^{t_2} \left(L(q_1, q_2, \dot q_1, \dot q_2, t) + \frac{\partial L}{\partial q_1}\delta q_1 + \frac{\partial L}{\partial q_2}\delta q_2 + \frac{\partial L}{\partial \dot q_1}\delta \dot q_1 + \frac{\partial L}{\partial \dot q_2}\delta \dot q_2 - L(q_1, q_2, \dot q_1, \dot q_2, t)\right) dt = 0$$

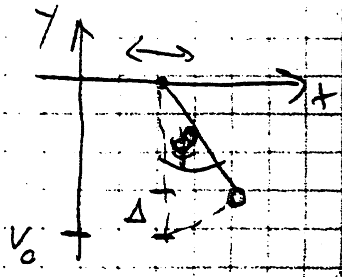
$$= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_1}\delta q_1 + \frac{\partial L}{\partial q_2}\delta q_2 + \frac{\partial L}{\partial \dot q_1}\delta \dot q_1 + \frac{\partial L}{\partial \dot q_2}\delta \dot q_2 \right) dt = 0$$

part. Integration: $\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot q}\delta \dot q\, dt = \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot q}\frac{d}{dt}\delta q = -\int_{t_1}^{t_2} \frac{d}{dt}\frac{\partial L}{\partial \dot q}\delta q + \underbrace{\frac{\partial L}{\partial \dot q}\delta q\Big|_{t_1}^{t_2}}_{=0}$

$$= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_1}\delta q_1 + \frac{\partial L}{\partial q_2}\delta q_2 - \frac{d}{dt}\frac{\partial L}{\partial \dot q_1}\delta q_1 - \frac{d}{dt}\frac{\partial L}{\partial \dot q_2}\delta q_2 \right) dt + \underbrace{\frac{\partial L}{\partial \dot q_1}\delta q_1\Big|_{t_1}^{t_2} + \frac{\partial L}{\partial \dot q_2}\delta q_2\Big|_{t_1}^{t_2}}_{=0} = 0$$

$$= \int_{t_1}^{t_2} \left(\left(\frac{\partial L}{\partial q_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot q_1}\right)\delta q_1 + \left(\frac{\partial L}{\partial q_2} - \frac{d}{dt}\frac{\partial L}{\partial \dot q_2}\right)\delta q_2 \right) dt = 0$$

Aufgabe P16   Aufhängepunkt: $x_0(t) = A\cdot\sin\Omega t$



Masse: $x(t) = x_0(t) + l\cdot\sin\varphi$

$y(t) = l\cdot\cos\varphi$

$\dot x(t) = A\Omega\cos\Omega t + l\dot\varphi\cos\varphi \qquad \dot y(t) = -l\dot\varphi\sin\varphi$

$$T = \frac{m}{2}v^2 = \frac{m}{2}(\dot x^2+\dot y^2) = \frac{m}{2}\Big(A^2\Omega^2\cos^2\Omega t + \underline{l^2\dot\varphi^2\cos^2\varphi} + \cancel{2A\Omega} +$$
$$+2Al\Omega\dot\varphi\cos\Omega t\cos\varphi + \underline{l^2\dot\varphi^2\sin^2\varphi}\Big) =$$
$$= \frac{m}{2}\left(A^2\Omega^2\cos^2\Omega t + l^2\dot\varphi^2 + 2Al\Omega\cos\Omega t\,\dot\varphi\cos\varphi\right)$$

$V = V_0 + mg\Delta \qquad \Delta = l - y(t) = l - l\cos\varphi = l(1-\cos\varphi)\,;\ V_0 = -mgl$

$V = -mgl + mgl(1-\cos\varphi) = -mgl\cos\varphi$

$$L = T - V = \frac{m}{2}\left(A^2\Omega^2\cos^2\Omega t + l^2\dot\varphi^2 + 2Al\Omega\cos\Omega t\,\dot\varphi\cos\varphi + 2gl\cos\varphi\right)$$

$$\frac{\partial L}{\partial\varphi} = m\left(\underline{-Al\Omega\cos\Omega t\,\dot\varphi\sin\varphi} - gl\sin\varphi\right)$$

$$\frac{\partial L}{\partial\dot\varphi} = ml^2\dot\varphi + mAl\Omega\cos\Omega t\cos\varphi$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\varphi} = ml^2\ddot\varphi - mAl\Omega^2\sin\Omega t\cos\varphi - \underline{mAl\Omega\cos\Omega t\sin\varphi\cdot\dot\varphi}$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\varphi} - \frac{\partial L}{\partial\varphi} = 0 \iff l^2\ddot\varphi - Al\Omega^2\sin\Omega t\cos\varphi + gl\sin\varphi = 0$$

$$\ddot\varphi + \frac{g}{l}\sin\varphi - \frac{A\Omega^2}{l}\sin\Omega t\cos\varphi = 0$$

Für kleine Auslenkungen gilt: $\sin\varphi\approx\varphi$, $\cos\varphi\approx 1$ $\Rightarrow \ddot\varphi + \frac{g}{l}\varphi = \frac{A\Omega^2}{l}\sin\Omega t$

Bekannte (inhomogene) Dgl. für erzwungene Schwingungen

Allg. Lsg. d. hom. Dgl: $\ddot\varphi + \frac{g}{l}\varphi = 0$ ist: $\varphi(t) = \varphi_0\cdot\sin(\omega_0 t+\delta)$

mit $\omega_0 = \sqrt{g/l}$

Spezielle Lsg. d. inhom. Dgl: Versuch: $\varphi = C\cdot\sin\Omega t \qquad \ddot\varphi = -C\Omega^2\sin\Omega t$

einsetzen: $-C\Omega^2\cancel{\sin}\,\Omega t + \frac{g}{l}C\cancel{\sin}\,\Omega t = \frac{A\Omega^2}{l}\cancel{\sin}\,\Omega t\,;\ C\left(\frac{g}{l}-\Omega^2\right) = \frac{A\Omega^2}{l}$

$C = \frac{A}{l}\frac{\Omega^2}{g/l-\Omega^2} = \frac{A}{l}\frac{\Omega^2}{\omega_0^2-\Omega^2}$; Anfangsbed.: $x(0)=0 \iff \delta = 0$

$\dot x(0) = 0 \iff \varphi_0 = -\frac{A}{l}\frac{\Omega}{\omega_0}\frac{\Omega^2}{\omega_0^2-\Omega^2}$

$\varphi(t) = \varphi_0\sin(\omega_0 t+\delta) + \frac{A}{l}\frac{\Omega^2}{\omega_0^2-\Omega^2}\sin\Omega t$

$$\varphi(t) = \frac{A}{l}\frac{\Omega^2}{\omega_0^2-\Omega^2}\left(\sin\Omega t - \frac{\Omega}{\omega_0}\sin\omega_0 t\right)$$

#18

a) Let $\mathbf{A} = \boldsymbol{\omega}$ in Problem P18 Then

$$\left.\frac{d\boldsymbol{\omega}}{dt}\right|_F = \left.\frac{d\boldsymbol{\omega}}{dt}\right|_M + \boldsymbol{\omega}\times\boldsymbol{\omega} = \left.\frac{d\boldsymbol{\omega}}{dt}\right|_M$$

Since $d\boldsymbol{\omega}/dt$ is the angular acceleration, the required statement is proved.

b)

Replacing $\mathbf{A}$ by the position vector $\mathbf{r}$ of the particle, we have

$$\left.\frac{d\mathbf{r}}{dt}\right|_F = \left.\frac{d\mathbf{r}}{dt}\right|_M + \boldsymbol{\omega}\times\mathbf{r} \tag{1}$$

If $\mathbf{r}$ is expressed in terms of the unit vectors $\mathbf{i},\mathbf{j},\mathbf{k}$ of the moving coordinate system, then the velocity of the particle relative to this system is, on dropping the subscript M,

$$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} \tag{2}$$

and the velocity of the particle relative to the fixed system is from (*1*)

$$\left.\frac{d\mathbf{r}}{dt}\right|_F = \frac{d\mathbf{r}}{dt} + \boldsymbol{\omega}\times\mathbf{r} \tag{3}$$

The velocity (*3*) is sometimes called the *true velocity,* while (*2*) is the *apparent velocity.*

c)

$$D_F \equiv D_M + \boldsymbol{\omega}\times$$

By definition

$$D_F\mathbf{A} = \left.\frac{d\mathbf{A}}{dt}\right|_F = \text{derivative in fixed system}$$

$$D_M\mathbf{A} = \left.\frac{d\mathbf{A}}{dt}\right|_M = \text{derivative in moving system}$$

Then from Problem P18

$$D_F\mathbf{A} = D_M\mathbf{A} + \boldsymbol{\omega}\times\mathbf{A} = (D_M + \boldsymbol{\omega}\times)\mathbf{A}$$

which shows the equivalence of the operators $D_F \equiv D_M + \boldsymbol{\omega}\times$.

The acceleration of the particle as seen by the observer in the fixed XYZ system is $D_F^2\mathbf{r} = D_F(D_F\mathbf{r})$. Using the operator equivalence we have

$$\begin{aligned} D_F(D_F\mathbf{r}) &= D_F(D_M\mathbf{r} + \boldsymbol{\omega}\times\mathbf{r}) \\ &= (D_M + \boldsymbol{\omega}\times\ )(D_M\mathbf{r} + \boldsymbol{\omega}\times\mathbf{r}) \\ &= D_M(D_M\mathbf{r} + \boldsymbol{\omega}\times\mathbf{r}) + \boldsymbol{\omega}\times(D_M\mathbf{r} + \boldsymbol{\omega}\times\mathbf{r}) \\ &= D_M^2\mathbf{r} + D_M(\boldsymbol{\omega}\times\mathbf{r}) + \boldsymbol{\omega}\times D_M\mathbf{r} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) \end{aligned}$$

or since $D_M(\boldsymbol{\omega}\times\mathbf{r}) = (D_M\boldsymbol{\omega})\times\mathbf{r} + \boldsymbol{\omega}\times(D_M\mathbf{r})$,

$$D_F^2\mathbf{r} = D_M^2\mathbf{r} + (D_M\boldsymbol{\omega})\times\mathbf{r} + 2\boldsymbol{\omega}\times(D_M\mathbf{r}) + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) \tag{1}$$

If $\mathbf{r}$ is the position vector expressed in terms of $\mathbf{i},\mathbf{j},\mathbf{k}$ of the moving coordinate system, then the acceleration of the particle relative to this system is, on dropping the subscript M,

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k} \tag{2}$$

The acceleration of the particle relative to the fixed system is given from (*1*) as

$$\left.\frac{d^2\mathbf{r}}{dt^2}\right|_F = \frac{d^2\mathbf{r}}{dt^2} + \frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r} + 2\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right) + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) \tag{3}$$

The acceleration (*3*) is sometimes called the *true acceleration,* while (*2*) is the *apparent acceleration.*

H19

(a) Inertial system B

$$\boxed{\begin{array}{lll} \ddot{x}_1 = 0 , & x_1(0) = 0 , & \dot{x}_1(0) = v_1 \\ \ddot{x}_2 = 0 , & x_2(0) = 0 , & \dot{x}_2(0) = v_2 \\ \ddot{x}_3 = -g , & x_3(0) = 0 , & \dot{x}_3(0) = v_3 \end{array}} \xrightarrow{\text{Integration}} \begin{array}{l} x_1(t) = v_1 t \\ x_2(t) = v_2 t \\ x_3(t) = v_3 t - \frac{g}{2} t^2 \end{array}$$

$$\left.\begin{array}{l} x_1' = \cos\omega t \, x_1 + \sin\omega t \, x_2 \\ x_2' = -\sin\omega t \, x_1 + \cos\omega t \, x_2 \\ x_3' = x_3 \end{array}\right\} \rightarrow \boxed{\begin{array}{l} x_1'(t) = v_1 t \cos\omega t + v_2 t \sin\omega t \\ x_2'(t) = -v_1 t \sin\omega t + v_2 t \cos\omega t \\ x_3'(t) = v_3 t - \frac{g}{2} t^2 \end{array}}$$

(b) Kartesisch

$$\boxed{\begin{array}{lll} \ddot{x}_1' = 2\omega \dot{x}_2' + \omega^2 x_1' & x_1'(0) = 0 , & \dot{x}_1'(0) = v_1 \\ \ddot{x}_2' = -2\omega \dot{x}_1' + \omega^2 x_2' & x_2'(0) = 0 , & \dot{x}_2'(0) = v_2 \end{array}} \quad \begin{array}{l} (B.b.1) \\ (B.b.2) \end{array}$$

$$u := x_1' + i x_2' \xrightarrow{(B.b.1) + i (B.b.2)} \boxed{\ddot{u} + 2 i \omega \dot{u} - \omega^2 u = 0}$$

Ansatz: $u \sim e^{\lambda t}$ liefert charakteristisches Polynom

$$\lambda^2 + 2 i \omega \lambda - \omega^2 = (\lambda + i\omega)^2 = 0$$

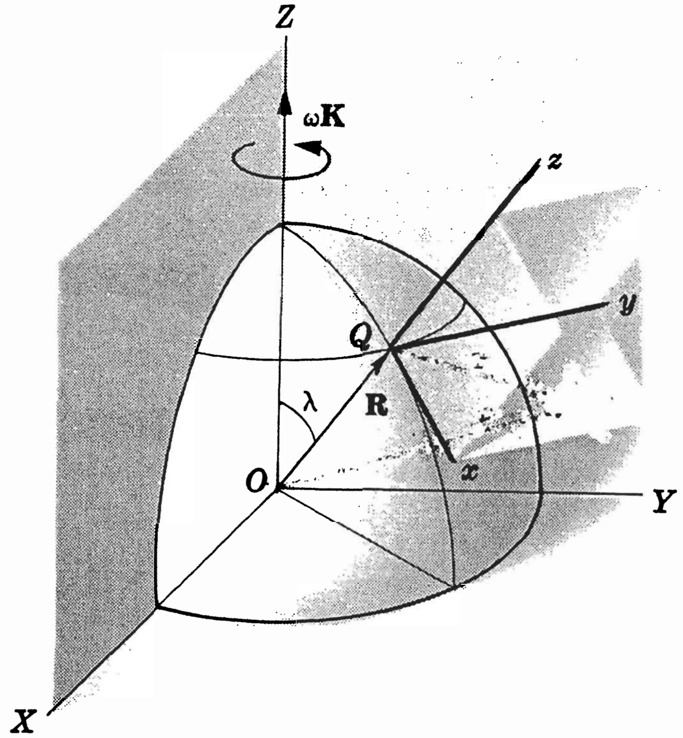
$$\rightarrow \lambda_{1/2} = -i\omega \quad (\text{Doppelwurzel})$$

$$\rightarrow u(t) = (A + Bt) e^{-i\omega t}$$

Anfangsbed. liefern $A = 0$, $B = v_1 + i v_2$

$$\boxed{\begin{array}{l} x_1'(t) = v_1 t \cos\omega t + v_2 t \sin\omega t = v t \cos(\varphi_0 - \omega t) \\ x_2'(t) = -v_1 t \sin\omega t + v_2 t \cos\omega t = v t \sin(\varphi_0 - \omega t) \end{array}} \quad \tan\varphi_0 = \frac{v_2}{v_1}$$

P17)



Im bewegten System: $\vec{\omega} = \omega \vec{e}_Z = -\omega \sin\lambda \, \vec{e}_x{}' + \omega \cos\lambda \, \vec{e}_z{}'$

$= -\omega \cos\theta \, \vec{e}_x{}' + \omega \sin\theta \, \vec{e}_z{}'$

$\left[\begin{array}{l} \sin(90°-\theta) = \cos(-\theta) = \cos\theta \\ \cos(90°-\theta) = -\sin(-\theta) = \sin\theta \end{array}\right.$

Corioliskraft: $\vec{F}_c = 2m \, \vec{v} \times \vec{\omega}$

a) Süd $\to$ Nord: $\vec{v} = -v\vec{e}_x{}'$

$\to \vec{F}_c = 2mv\omega \sin\theta \, \vec{e}_y{}'$ nach Osten gerichtet

b) Ost $\to$ West: $\vec{v} = -v\vec{e}_y{}'$

$\vec{F}_c = -2mv\vec{e}_y{}' \times \left[-\omega\cos\theta \, \vec{e}_x{}' + \omega \sin\theta \, \vec{e}_z{}'\right]$

$= -2mv\omega \left(\underbrace{\vec{e}_x{}' \times \vec{e}_y{}'}_{= \vec{e}_z{}'} \cdot \cos\theta + \underbrace{\vec{e}_y{}' \times \vec{e}_z{}'}_{= \vec{e}_x{}'} \sin\theta \right)$

$= -2mv\omega \left(\cos\theta \, \vec{e}_z{}' + \sin\theta \, \vec{e}_x{}' \right)$

nach Süden gerichtet

P18

To the fixed observer the unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ actually change with time. Hence such an observer would compute the time derivative as

$$\frac{d\mathbf{A}}{dt} = \frac{dA_1}{dt}\mathbf{i} + \frac{dA_2}{dt}\mathbf{j} + \frac{dA_3}{dt}\mathbf{k} + A_1\frac{d\mathbf{i}}{dt} + A_2\frac{d\mathbf{j}}{dt} + A_3\frac{d\mathbf{k}}{dt} \tag{1}$$

i.e.,

$$\left.\frac{d\mathbf{A}}{dt}\right|_F = \left.\frac{d\mathbf{A}}{dt}\right|_M + A_1\frac{d\mathbf{i}}{dt} + A_2\frac{d\mathbf{j}}{dt} + A_3\frac{d\mathbf{k}}{dt} \tag{2}$$

Since $\mathbf{i}$ is a unit vector, $d\mathbf{i}/dt$ is perpendicular to $\mathbf{i}$ and must therefore lie in the plane of $\mathbf{j}$ and $\mathbf{k}$. Then

$$d\mathbf{i}/dt = \alpha_1\mathbf{j} + \alpha_2\mathbf{k} \tag{3}$$

Similarly,

$$d\mathbf{j}/dt = \alpha_3\mathbf{k} + \alpha_4\mathbf{i} \tag{4}$$

$$d\mathbf{k}/dt = \alpha_5\mathbf{i} + \alpha_6\mathbf{j} \tag{5}$$

From $\mathbf{i}\cdot\mathbf{j} = 0$, differentiation yields $\mathbf{i}\cdot\frac{d\mathbf{j}}{dt} + \frac{d\mathbf{i}}{dt}\cdot\mathbf{j} = 0$. But $\mathbf{i}\cdot\frac{d\mathbf{j}}{dt} = \alpha_4$ from (4) and $\frac{d\mathbf{i}}{dt}\cdot\mathbf{j} = \alpha_1$ from (3). Thus $\alpha_4 = -\alpha_1$.

Similarly from $\mathbf{i}\cdot\mathbf{k} = 0$, $\mathbf{i}\cdot\frac{d\mathbf{k}}{dt} + \frac{d\mathbf{i}}{dt}\cdot\mathbf{k} = 0$ and $\alpha_5 = -\alpha_2$; from $\mathbf{j}\cdot\mathbf{k} = 0$, $\mathbf{j}\cdot\frac{d\mathbf{k}}{dt} + \frac{d\mathbf{j}}{dt}\cdot\mathbf{k} = 0$ and $\alpha_6 = -\alpha_3$. Then

$$d\mathbf{i}/dt = \alpha_1\mathbf{j} + \alpha_2\mathbf{k}, \quad d\mathbf{j}/dt = \alpha_3\mathbf{k} - \alpha_1\mathbf{i}, \quad d\mathbf{k}/dt = -\alpha_2\mathbf{i} - \alpha_3\mathbf{j}$$

It follows that

$$A_1\frac{d\mathbf{i}}{dt} + A_2\frac{d\mathbf{j}}{dt} + A_3\frac{d\mathbf{k}}{dt} = (-\alpha_1A_2 - \alpha_2A_3)\mathbf{i} + (\alpha_1A_1 - \alpha_3A_3)\mathbf{j} + (\alpha_2A_1 + \alpha_3A_2)\mathbf{k} \tag{6}$$

which can be written as

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \alpha_3 & -\alpha_2 & \alpha_1 \\ A_1 & A_2 & A_3 \end{vmatrix}$$

Then if we choose $\alpha_3 = \omega_1$, $-\alpha_2 = \omega_2$, $\alpha_1 = \omega_3$ this determinant becomes

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_1 & \omega_2 & \omega_3 \\ A_1 & A_2 & A_3 \end{vmatrix} = \boldsymbol{\omega}\times\mathbf{A}$$

where $\boldsymbol{\omega} = \omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k}$.

From (2) and (6) we find, as required,

$$\left.\frac{d\mathbf{A}}{dt}\right|_F = \left.\frac{d\mathbf{A}}{dt}\right|_M + \boldsymbol{\omega}\times\mathbf{A}$$

The vector quantity $\boldsymbol{\omega}$ is the *angular velocity* of the moving system relative to the fixed system.

P19

(a) The apparent velocity at any time t is

$$d\mathbf{r}/dt = 2t\mathbf{i} - 6\mathbf{j} + 12t^2\mathbf{k}$$

At time $t = 1$ this is $2\mathbf{i} - 6\mathbf{j} + 12\mathbf{k}$.

The true velocity at any time t is

$$d\mathbf{r}/dt + \boldsymbol{\omega} \times \mathbf{r} = (2t\mathbf{i} - 6\mathbf{j} + 12t^2\mathbf{k}) + [2t\mathbf{i} - t^2\mathbf{j} + (2t+4)\mathbf{k}] \times [(t^2+1)\mathbf{i} - 6t\mathbf{j} + 4t^3\mathbf{k}]$$

At time $t = 1$ this is

$$2\mathbf{i} - 6\mathbf{j} + 12\mathbf{k} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 6 \\ 2 & -6 & 4 \end{vmatrix} = 34\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$$

(b) The apparent acceleration at any time t is

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right) = \frac{d}{dt}(2t\mathbf{i} - 6\mathbf{j} + 12t^2\mathbf{k}) = 2\mathbf{i} + 24t\mathbf{k}$$

At time $t = 1$ this is $2\mathbf{i} + 24\mathbf{k}$.

The true acceleration at any time t is

$$\frac{d^2\mathbf{r}}{dt^2} + 2\boldsymbol{\omega} \times \frac{d\mathbf{r}}{dt} + \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

At time $t = 1$ this equals

$$\begin{aligned} & 2\mathbf{i} + 24\mathbf{k} + (4\mathbf{i} - 2\mathbf{j} + 12\mathbf{k}) \times (2\mathbf{i} - 6\mathbf{j} + 12\mathbf{k}) \\ & \quad + (2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) \times (2\mathbf{i} - 6\mathbf{j} + 4\mathbf{k}) \\ & \quad\quad + (2\mathbf{i} - \mathbf{j} + 6\mathbf{k}) \times \{(2\mathbf{i} - \mathbf{j} + 6\mathbf{k}) \times (2\mathbf{i} - 6\mathbf{j} + 4\mathbf{k})\} \\ = {} & 2\mathbf{i} + 24\mathbf{k} + (48\mathbf{i} - 24\mathbf{j} - 20\mathbf{k}) + (4\mathbf{i} - 4\mathbf{j} - 8\mathbf{k}) + (-14\mathbf{i} + 212\mathbf{j} + 40\mathbf{k}) \\ = {} & 40\mathbf{i} + 184\mathbf{j} + 36\mathbf{k} \end{aligned}$$

(c) From Problem (a) we have,

$$\text{Coriolis acceleration} = 2\boldsymbol{\omega} \times d\mathbf{r}/dt = (4\mathbf{i} - 2\mathbf{j} + 12\mathbf{k}) \times (2\mathbf{i} - 6\mathbf{j} + 12\mathbf{k}) = 48\mathbf{i} - 24\mathbf{j} - 20\mathbf{k}$$

From Problem (a) we have,

$$\text{Centripetal acceleration} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = (2\mathbf{i} - \mathbf{j} + 6\mathbf{k}) \times (32\mathbf{i} + 4\mathbf{j} - 10\mathbf{k}) = -14\mathbf{i} + 212\mathbf{j} + 40\mathbf{k}$$

$$\text{Magnitude of Coriolis acceleration} = \sqrt{(48)^2 + (-24)^2 + (-20)^2} = 4\sqrt{205}$$

$$\text{Magnitude of centripetal acceleration} = \sqrt{(-14)^2 + (212)^2 + (40)^2} = 2\sqrt{11{,}685}$$

P20

Choose the xyz coordinate system of Fig. 1 Suppose that the origin O is the equilibrium position of the bob B, A is the point of suspension and the length of string AB is l. If the tension in the string is $\mathbf{T}$, then we have

$$\begin{aligned} \mathbf{T} &= (\mathbf{T}\cdot\mathbf{i})\mathbf{i} + (\mathbf{T}\cdot\mathbf{j})\mathbf{j} + (\mathbf{T}\cdot\mathbf{k})\mathbf{k} \\ &= T\cos\alpha\,\mathbf{i} + T\cos\beta\,\mathbf{j} + T\cos\gamma\,\mathbf{k} \\ &= -T\left(\frac{x}{l}\right)\mathbf{i} - T\left(\frac{y}{l}\right)\mathbf{j} + T\left(\frac{l-z}{l}\right)\mathbf{k} \qquad (1) \end{aligned}$$

Since the net force acting on B is $\mathbf{T} + m\mathbf{g}$, the equation of motion of B is given by [see Problem 6.14]

$$m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{T} + m\mathbf{g} - 2m(\boldsymbol{\omega}\times\mathbf{v}) - m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) \qquad (2)$$

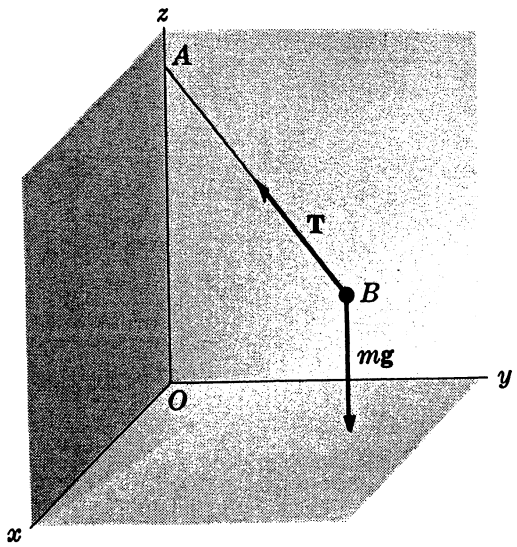


Fig 1

If we neglect the last term in (2), put $\mathbf{g} = -g\mathbf{k}$ and use (1), then (2) can be written in component form as

$$m\ddot{x} = -T(x/l) + 2m\omega\dot{y}\cos\lambda \qquad (3)$$

$$m\ddot{y} = -T(y/l) - 2m\omega(\dot{x}\cos\lambda + \dot{z}\sin\lambda) \qquad (4)$$

$$m\ddot{z} = T(l-z)/l - mg + 2m\omega\dot{y}\sin\lambda \qquad (5)$$

P21

Eine Perle gleitet auf einem geraden Draht, der mit konstanter Winkelgeschwindigkeit ω in der horizontalen Ebene rotiert. Stelle die Hamiltonfunktion auf und berechne $r(t)$.

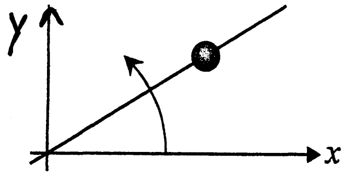


Abb. 15.3–2 Perle rutscht auf rotierendem Draht.

Lösung:

Die Zwangsbedingung ist rheonom. Wegen $V = 0$ gilt:

$$L = T = \frac{m}{2}(\dot{r}^2 + r^2\,\omega^2) = E_{\text{Perle}}$$

$$\Rightarrow \quad H = \frac{p_r^2}{2m} - \frac{m}{2} r^2\,\omega^2 \neq E_{\text{Perle}}$$

Wir sehen, daß Lagrange- und Hamiltonfunktion bei rheonomen Zwangsbedingungen zeitunabhängig sein können. Die allgemeine Lösung $r(t)$ der Bewegungsgl. $\ddot{r} = \omega^2\,r$ lautet:

$$r(t) = c_1\,e^{\omega t} + c_2\,e^{-\omega t}$$

wobei c_1, c_2 die beiden Integrationskonstanten sind. Einsetzen von $r(t)$ und $p_r(t) = m\,\dot{r}(t)$ in die Hamiltonfunktion liefert nach kurzer Rechnung:

$$H = \text{const}$$

Dieses Ergebnis ist natürlich auch direkt – ohne Rechnung – aus der expliziten Zeitunabhängigkeit von L bzw. H zu entnehmen.

P22

Berechne die eindimensionalen Schwingungen eines dreiatomigen, linearen Moleküls mit zwei gleichen Federkonstanten D.

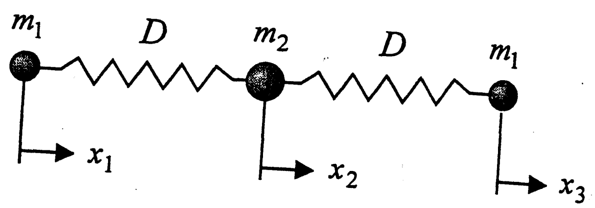


Abb. 15.3–1 Dreiatomiges Molekül

Lösung:

Die Hamiltonfunktion des Moleküls lautet

$$H = T + V = \frac{1}{2m_1}\left(p_1^2 + p_3^2\right) + \frac{1}{2m_2}p_2^2 + \frac{D}{2}\left[(x_1 - x_2)^2 + (x_3 - x_2)^2\right]$$

mit x_i = die Verrückungen aus der Gleichgewichtslage. Die kanonischen Gln. lauten:

$$\dot{x}_1 = \frac{\partial H}{\partial p_1} = \frac{p_1}{m_1} \qquad \dot{p}_1 = -\frac{\partial H}{\partial x_1} = -D\,(x_1 - x_2)$$

$$\Rightarrow \quad m_1\,\ddot{x}_1 = -D\,(x_1 - x_2)$$

$$\dot{x}_2 = \frac{\partial H}{\partial p_2} = \frac{p_2}{m_2} \qquad \dot{p}_2 = -\frac{\partial H}{\partial x_2} = -D\left[(x_2 - x_1) + (x_2 - x_3)\right]$$

$$\Rightarrow \quad m_2\,\ddot{x}_2 = -D\,(2\,x_2 - x_1 - x_3)$$

Auf die gleiche Weise erhält man die dritte Bewegungsgl.:

$$m_1\,x_3 = -D\,(x_3 - x_2)$$