8 Hausaufgabe zu Theorie der höheren Mechanik zum Montag, den 14.6.2010

H14

$L = T - U = T_{tran} + T_{rot} - U$

$T_{tran} = \frac{M}{2} \sum_{k=1}^{3} \dot{X}_k^2 = T_{tran}(\dot{X}_1, \dot{X}_2, \dot{X}_3)$

$T_{rot} = \frac{1}{2} \sum_{k=1}^{3} I_k \omega_k^2 = T_{rot}(\phi, \psi, \theta, \dot{\phi}, \dot{\psi}, \dot{\theta})$

$\Rightarrow U = U(X_1, X_2, X_3, \phi, \psi, \theta)$

Also gilt für alle k: $\frac{d}{dt}(\frac{\partial T}{\partial \dot{X}_k}) - \frac{\partial T}{\partial X_k} = -\frac{\partial U}{\partial X_k} \Rightarrow M\ddot{X}_k = -\frac{\partial U}{\partial X_k}$

$\frac{d}{dt}(\frac{\partial T_{rot}}{\partial \dot{\psi}}) - \frac{\partial T_{rot}}{\partial \psi} = -\frac{\partial U}{\partial \psi} = \frac{d}{dt} \sum_k \frac{\partial T_{rot}}{\partial \omega_k} \frac{\partial \omega_k}{\partial \dot{\psi}} - \sum_k \frac{\partial T_{rot}}{\partial \omega_k} \frac{\partial \omega_k}{\partial \psi}$

$\frac{\partial \omega_1}{\partial \dot{\psi}} = \frac{\partial \omega_2}{\partial \dot{\psi}} = 0, \quad \frac{\partial \omega_3}{\partial \dot{\psi}} = 1$

$\frac{\partial \omega_1}{\partial \psi} = \dot{\phi} \sin(\theta) \cos(\psi) - \dot{\theta} \sin(\psi) = \omega_2, \quad \frac{\partial \omega_2}{\partial \psi} = -\dot{\phi} \sin(\theta) \cos(\psi) - \dot{\theta} \cos(\psi) = -\omega_1, \quad \frac{\partial \omega_3}{\partial \psi} = 0$

$\Rightarrow \frac{d}{dt}(\frac{\partial T_{rot}}{\partial \dot{\psi}}) - \frac{\partial T_{rot}}{\partial \psi} = \frac{d}{dt}(I_3 \omega_3) - (I_1 \omega_1 \omega_2 - I_2 \omega_2 \omega_1) = I_3 \dot{\omega}_3 + (I_2 - I_1) \omega_1 \omega_2 = -\frac{\partial U}{\partial \psi}$

$M(t) = \sum_{k=1}^{N} r_k(t) \times F_k$

$$\Rightarrow \left| \begin{array}{l} I_1 \dot{\omega}_1 + (I_3 - I_2) \omega_2 \omega_3 = M_1(t) \\ I_2 \dot{\omega}_2 + (I_1 - I_3) \omega_1 \omega_3 = M_2(t) \\ I_3 \dot{\omega}_3 + (I_2 - I_1) \omega_1 \omega_2 = M_3(t) \end{array} \right|$$

H15

a) Zykloide: $x = a(\theta - \sin(\theta)), \quad y = a(1 + \cos(\theta))$ $(0 \leq \theta \leq 2\pi)$

kinetische Energie:

$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m a^2 \dot{\theta}^2 ((1 - \cos(\theta))^2 + \sin^2(\theta)) = m a^2 (1 - \cos(\theta)) \dot{\theta}^2$

Potentialle Energie:

$V = mgy = mga(1 + \cos(\theta))$

$\Rightarrow$ Lagrangefunktion:

$L = T - V = m a^2 (1 - \cos(\theta)) \dot{\theta}^2 - mga(1 + \cos(\theta))$

b) $\frac{d}{dt} \frac{\partial L}{\partial \dot{\vartheta}} - \frac{\partial L}{\partial \theta} = \frac{d}{dt}(2ma^2(1 - \cos(\theta)\dot{\theta})) - (ma^2 \sin(\theta) \dot{\theta}^2 + mga \sin(\theta)) = 0$

$\Rightarrow \frac{d}{dt}((1 - cos(\theta))\dot{\theta}) - \frac{1}{2} \sin(\theta) \dot{\theta}^2 - \frac{g}{2a} \sin(\theta) = (1 - \cos(\theta)) \ddot{\theta} + \frac{1}{2} \sin(\theta) \dot{\theta}^2 - \frac{g}{2a} \sin(\theta) = 0$

$\ddot{\theta} + \frac{1}{2} \frac{\sin(\theta)}{(1 - \cos(\theta))} \dot{\theta}^2 - \frac{g}{2a} \frac{\sin(\theta)}{(1 - \cos(\theta))} = 0$

c) $u = \cos(\frac{\theta}{2}) \Rightarrow \frac{du}{dt} = -\frac{1}{2} \sin(\frac{\theta}{2}) \dot{\theta}$

$\Rightarrow \frac{d^2 u}{dt^2} = -\frac{1}{2} \sin(\frac{\theta}{2}) \ddot{\theta} - \frac{1}{4} \cos(\frac{\theta}{2}) \dot{\theta}^2$

$(1 - \cos(\theta)) \ddot{\theta} + \frac{1}{2} \sin(\theta) \dot{\theta}^2 - \frac{g}{2a} \sin(\theta) = 0$

$\Rightarrow \ddot{\theta} + \frac{1}{2} \frac{\sin(\theta)}{(1 - \cos(\theta))} \dot{\theta}^2 - \frac{g}{2a} \frac{\sin(\theta)}{(1 - \cos(\theta))} = \ddot{\theta} + \frac{1}{2} \cot(\frac{\theta}{2}) \dot{\theta}^2 - \frac{g}{2a} \cot(\frac{\theta}{2}) = \ddot{\theta} + \frac{1}{2} \frac{\cos(\frac{\theta}{2})}{\sin(\frac{\theta}{2})} \dot{\theta}^2 - \frac{g}{2a} \frac{\cos(\frac{\theta}{2})}{\sin(\frac{\theta}{2})} = 0$

$\Rightarrow \frac{d^2 u}{dt^2} + \frac{g}{4a} u = 0$

$\Rightarrow u = \cos(\frac{\theta}{2}) = C_1 \cos(\sqrt{\frac{g}{4a}} t) + C_2 \sin(\sqrt{\frac{g}{4a}} t)$

d) Mit $t = 2\pi\sqrt{\frac{4a}{g}}k$ $(k \in \mathbb{N})$ ist $u = C_1 \cos(\sqrt{\frac{g}{4a}} \cdot 2\pi\sqrt{\frac{4a}{g}}k) + C_2 \sin(\sqrt{\frac{g}{4a}} \cdot 2\pi\sqrt{\frac{4a}{g}}k)$

$= C_1 \cos(2\pi \cdot k) + C_2 \sin(2\pi \cdot k)$

Da Sinus und Kosinus mit Periode 2π schwingen, osziliert die Perle mit $2\pi\sqrt{\frac{4a}{g}}$