9 Hausaufgabe zu Theorie der höheren Mechanik zum Montag, den 21.6.2010

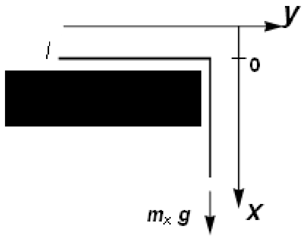
H16

a) $T = \frac{1}{2}m(\dot{y}^2 + \dot{x}^2) = \frac{1}{2}\varrho x\dot{x}^2 + \frac{1}{2}\varrho(l-x)(\dot{\overline{l-x}})^2 = \frac{1}{2}\varrho(x\dot{x}^2 + (l-x)(-\dot{x})^2$

$= \frac{1}{2}\varrho\dot{x}^2(x + l - x) = \frac{1}{2}\varrho\dot{x}^2 l$

$V = m_x g\ dx = \varrho x g\ dx = \frac{1}{2}\varrho g x^2$

$\Rightarrow L = T - V = \frac{1}{2}\varrho\dot{x}^2 l - \frac{1}{2}\varrho g x^2$



b) $\frac{d}{dt}\frac{\partial V}{\partial \dot{x}} - \frac{\partial V}{\partial x} = 0 \quad \frac{\partial V}{\partial \dot{x}} = \varrho\dot{x}l$

$\frac{d}{dt}\frac{\partial V}{\partial \dot{x}} = \varrho\ddot{x}l$

$\frac{\partial V}{\partial x} = \varrho g x$

$\Rightarrow \varrho\ddot{x}l - \varrho g x = 0 \Rightarrow \ddot{x} - \frac{g}{l}x = 0 \Rightarrow \omega = \pm\sqrt{\frac{g}{l}} \quad \Rightarrow x(t) = Ae^{\sqrt{\frac{g}{l}}t} + Be^{-\sqrt{\frac{g}{l}}t}$

c) $\dot{x}(0) = A\sqrt{\frac{g}{l}}e^{\sqrt{\frac{g}{l}}\cdot 0} - B\sqrt{\frac{g}{l}}e^{-\sqrt{\frac{g}{l}}\cdot 0} = 0 \Rightarrow A - B = 0 \Rightarrow A = B$

$x(0) = Ae^{\sqrt{\frac{g}{l}}\cdot 0} + Ae^{-\sqrt{\frac{g}{l}}\cdot 0} = 2A = a \Rightarrow A = \frac{a}{2}$

$\Rightarrow x(t) = \frac{a}{2}(e^{\sqrt{\frac{g}{l}}t} + e^{-\sqrt{\frac{g}{l}}t}) = a\cosh(\sqrt{\frac{g}{l}}t)$

9.1

a) $q_i = x_i - x_{i0}, \quad i = 1,2,3$

$T = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2)$

$V = \frac{1}{2}k((x_1 - x_{10})^2 + ((x_1 - x_{10}) - (x_2 - x_{20}))^2 + ((x_2 - x_{20}) - (x_3 - x_{30}))^2 + (x_3 - x_{30})^2) =$

$\frac{1}{2}k(q_1^2 + (q_1 - q_2)^2 + (q_2 - q_3)^2 + q_3^2)$

$\Rightarrow L = T - V = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) - \frac{1}{2}k(q_1^2 + (q_1 - q_2)^2 + (q_2 - q_3)^2 + q_3^2)$

$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_1}) - \frac{\partial L}{\partial q_1} = m\ddot{q}_1 + kq_1 + k(q_1 - q_2) = m\ddot{q}_1 + 2kq_1 - kq_2$

$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_2}) - \frac{\partial L}{\partial q_2} = m\ddot{q}_1 + k(q_2 - q_1) + k(q_2 - q_3) = m\ddot{q}_2 + 2kq_2 - kq_1 - kq_3$

$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_3}) - \frac{\partial L}{\partial q_3} = m\ddot{q}_3 + kq_1 + k(q_3 - q_2) = m\ddot{q}_3 + 2kq_3 - kq_2$

b) $\Rightarrow m\begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{pmatrix} + k\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\det\begin{pmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{pmatrix} = (-2+\lambda) + (2-\lambda)^3 - (2-\lambda) = -2(2-\lambda) + (2-\lambda)^3 \overset{!}{=} 0$

$\Rightarrow \lambda_1 = 2 \Rightarrow -2 + (2-\lambda)^2 = 0 \Rightarrow 2 - 4\lambda + \lambda^2 = 0 \Rightarrow \lambda_{2,3} = 2 \pm \sqrt{2}$

$\Rightarrow \omega_1^2 = 2\frac{k}{m}, \quad \omega_2^2 = (2+\sqrt{2})\frac{k}{m}, \quad \omega_3^2 = (2-\sqrt{2})\frac{k}{m}$

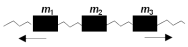
$\Rightarrow \omega_1 = \sqrt{2\frac{k}{m}}, \quad \omega_2 = \sqrt{(2+\sqrt{2})\frac{k}{m}}, \quad \omega_3 = \sqrt{(2-\sqrt{2})\frac{k}{m}}$

mit Ansatz: $q_j = A_j e^{i\omega_j t}, \quad j = 1,2,3$

c) $\begin{pmatrix} 2-\lambda_i & -1 & 0 \\ -1 & 2-\lambda_i & -1 \\ 0 & -1 & 2-\lambda_i \end{pmatrix}\begin{pmatrix} v_1^i \\ v_2^i \\ v_3^i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 2-2 & -1 & 0 \\ -1 & 2-2 & -1 \\ 0 & -1 & 2-2 \end{pmatrix} \begin{pmatrix} v_1^1 \\ v_2^1 \\ v_3^1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \left| \begin{array}{ll} -v_2^1 & = 0 \\ -v_1^1 - v_3^1 & = 0 \end{array} \right| \Rightarrow v^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

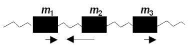
(m_2 ruht, m_1 bewegt sich entgegengesetzt mit gleicher Amplitude zu m_3.)



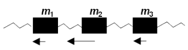
$$\Rightarrow \begin{pmatrix} 2-(2+\sqrt{2}) & -1 & 0 \\ -1 & 2-(2+\sqrt{2}) & -1 \\ 0 & -1 & 2-(2+\sqrt{2}) \end{pmatrix} \begin{pmatrix} v_1^2 \\ v_2^2 \\ v_3^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left| \begin{array}{ll} -\sqrt{2}v_1^2 - v_2^2 & = 0 \\ -v_1^2 - \sqrt{2}v_2^2 - v_3^2 & = 0 \\ -v_2^2 - \sqrt{2}v_3^2 & = 0 \end{array} \right| \Rightarrow \left| \begin{array}{ll} v_1^2 & = -\frac{1}{\sqrt{2}}v_2^2 \\ (\sqrt{2} - \frac{2}{\sqrt{2}})v_2^2 & = 0 \\ v_3^2 & = -\frac{1}{\sqrt{2}}v_2^2 \end{array} \right| \Rightarrow v^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -1 \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0.5 \\ -\frac{1}{\sqrt{2}} \\ 0.5 \end{pmatrix}$$

(m_1 und m_3 bewegen sich in die selbe Richtung mit gleicher Amplitude, m_2 bewegt sich entgegengesetzt mit größerer Amplitude Faktor $\sqrt{2}$.)



$$\Rightarrow \begin{pmatrix} 2-(2-\sqrt{2}) & -1 & 0 \\ -1 & 2-(2-\sqrt{2}) & -1 \\ 0 & -1 & 2-(2-\sqrt{2}) \end{pmatrix} \begin{pmatrix} v_1^3 \\ v_2^3 \\ v_3^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow v^3 = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0.5 \\ \frac{1}{\sqrt{2}} \\ 0.5 \end{pmatrix}$$



(siehe v^2, m_2 bewegt sich nun jedoch in die gleiche Richtung wie m_1, m_3)

d) $$A^T A = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}+\frac{1}{2} & \frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} & \frac{1}{2}-\frac{1}{2} \\ \frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} & \frac{1}{2}+\frac{1}{2} & \frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} \\ \frac{1}{2}-\frac{1}{2} & \frac{1}{2}-\frac{1}{2} & \frac{1}{2}+\frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

e) $T = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) = \frac{1}{2}\dot{\vec{q}}^T \mathcal{T} \dot{\vec{q}} \Rightarrow \mathcal{T} = m \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$V = \frac{1}{2}k(q_1^2 + (q_1 - q_2)^2 + (q_2 - q_3)^2 + q_3^2) = \frac{1}{2}k(2q1^2 - 2q1q2 + 2q2^2 - 2q2q3 + 2q3^2) = \frac{1}{2}\vec{q}^T \mathcal{V} \vec{q} =$
$\frac{1}{2}\vec{q}^T \begin{pmatrix} v_1 & v_2 & v_3 \\ v_4 & v_5 & v_6 \\ v_7 & v_8 & v_9 \end{pmatrix} \vec{q}$

$= v_1 q_1^2 + v_2 q_1 q_2 + v_3 q_1 q_3 + v_4 q_1 q_2 + v_5 q_2^2 + v_6 q_2 q_3 + v_7 q_1 q_3 + v_8 q_2 q_3 + v_9 q_3^2$

$\Rightarrow v_1 = 2k,\ v_2 = -k,\ v_3 = 0,\ v_4 = -k,\ v_5 = 2k,\ v_6 = -k,\ v_7 = 0,\ v_8 = -k,\ v_9 = 2k$

$$\Rightarrow \mathcal{V} = \begin{pmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{pmatrix}$$

f) $$(\mathcal{V} - \omega^2 \mathcal{T})\vec{a} = \vec{0} \Rightarrow \begin{pmatrix} 2k - m\omega^2 & -k & 0 \\ -k & 2k - m\omega^2 & -k \\ ß & -k & 2k - m\omega^2 \end{pmatrix} = k \begin{pmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{pmatrix} = \vec{0}$$

(gleich, siehe b)!)

g) $$A^T \mathcal{V} A = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$= k \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{2}} & 1+\frac{1}{\sqrt{2}} & 1-\frac{1}{\sqrt{2}} \\ 0 & -1-\frac{2}{\sqrt{2}} & -1+\frac{2}{\sqrt{2}} \\ -\frac{2}{\sqrt{2}} & 1+\frac{1}{\sqrt{2}} & 1-\frac{1}{\sqrt{2}} \end{pmatrix} =$$

$$\begin{pmatrix} 2 & \frac{1}{\sqrt{2}} + \frac{1}{2} - \frac{1}{\sqrt{2}} - \frac{1}{2} & \frac{1}{\sqrt{2}} - \frac{1}{2} - \frac{1}{\sqrt{2}} + \frac{1}{2} \\ \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} & \frac{1}{2} + \frac{1}{2\sqrt{2}} + \frac{1}{\sqrt{2}} + 1 + \frac{1}{2} + \frac{1}{2\sqrt{2}} & \frac{1}{2} - \frac{1}{2\sqrt{2}} + \frac{1}{2} - \frac{1}{2\sqrt{2}} + \frac{1}{\sqrt{2}} - 1 \\ \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} & \frac{1}{2} + \frac{1}{2\sqrt{2}} - \frac{1}{\sqrt{2}} - 1 + \frac{1}{2} + \frac{1}{2\sqrt{2}} & \frac{1}{2} - \frac{1}{2\sqrt{2}} - \frac{1}{2} + \frac{1}{2\sqrt{2}} + \frac{1}{\sqrt{2}} - 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2+\sqrt{2} & 0 \\ 0 & 0 & 2-\sqrt{2} \end{pmatrix}$$

h) $X = \begin{pmatrix} \frac{1}{\sqrt{2}}q_1 - \frac{1}{\sqrt{2}}q_3 \\ \frac{1}{2}q_1 - \frac{1}{\sqrt{2}}q_2 + \frac{1}{2}q_3 \\ -\frac{1}{2}q_1 + \frac{1}{\sqrt{2}}q_2 + \frac{1}{2}q_3 \end{pmatrix}$

Aus a) folgt: $\ddot{q}_1 = \frac{k}{m}(-2q_1 + q_2), \quad \ddot{q}_2 = \frac{k}{m}(-2q_2 + q_1 + q_3), \quad \ddot{q}_3 = \frac{k}{m}(-2q_3 + q_2)$

$\ddot{X}_1 = \frac{1}{\sqrt{2}}(\ddot{q}_1 - \ddot{q}_3) = \frac{1}{\sqrt{2}}\frac{k}{m}(-2q_1 + q_2 + 2q_3 - q_2)$

$= -\frac{1}{\sqrt{2}}\frac{k}{m}(2q_1 - 2q_3) = -\sqrt{2}\frac{k}{m}X_1 = -\omega_1^2 X_1$

$\ddot{X}_2 = \frac{1}{2}q_1 - \frac{1}{\sqrt{2}}q_2 + \frac{1}{2}q_3 = \frac{k}{m}(-q_1 + \frac{1}{2}q_2 - \frac{1}{\sqrt{2}}q_1 + \frac{2}{\sqrt{2}}q_2 - \frac{1}{\sqrt{2}}q_3 + \frac{1}{2}q_2 - q_3)$

$= \frac{k}{m}(q_1(-1 - \frac{1}{\sqrt{2}}) + q_2(1 + \sqrt{2}) + q_3(-1 - \frac{1}{\sqrt{2}}))$

$= -\frac{k}{m}(\frac{1}{2}q_1(2 + \sqrt{2}) - \frac{1}{\sqrt{2}}q_2(2 + \sqrt{2}) + \frac{1}{2}q_3(2 + \sqrt{2}))$

$= -(2 + \sqrt{2})\frac{k}{m}X_2 = -\omega_2^2 X_2$

$\ddot{X}_3 = \frac{1}{2}q_1 + \frac{1}{\sqrt{2}}q_2 + \frac{1}{2}q_3 = \frac{k}{m}(-q_1 + \frac{1}{2}q_2 + \frac{1}{\sqrt{2}}q_1 - \frac{2}{\sqrt{2}}q_2 + \frac{1}{\sqrt{2}}q_3 + \frac{1}{2}q_2 - q_3)$

$= \frac{k}{m}(q_1(-1 + \frac{1}{\sqrt{2}}) + q_2(1 - \sqrt{2}) + q_3(-1 + \frac{1}{\sqrt{2}}))$

$= -\frac{k}{m}(\frac{1}{2}q_1(2 - \sqrt{2}) - \frac{1}{\sqrt{2}}q_2(2 - \sqrt{2}) + \frac{1}{2}q_3(2 - \sqrt{2}))$

$= -(2 - \sqrt{2})\frac{k}{m}X_3 = -\omega_3^2 X_3$

$\Rightarrow \ddot{X}_1 + \omega_1^2 X_1 = 0, \quad \ddot{X}_2 + \omega_2^2 X_2 = 0, \quad \ddot{X}_3 + \omega_3^2 X_3 = 0$

Nun sind die Schwingungsgleichungen ungekoppelt.