6 Hausaufgabe zu Theorie der höheren Mechanik zum Montag, den 31.5.2010

H11

a) $I_{ij} = \int \sigma_0 (r^2 \delta ij - x_i x_j) dF$

$$I_1 = \begin{pmatrix} \sigma_0 \int\limits_{-a/2}^{a/2} \int\limits_{-b/2}^{b/2} (y^2+z^2) dxdy & -\sigma_0 \int\limits_{-a/2}^{a/2} \int\limits_{-b/2}^{b/2} xydxdy & -\sigma_0 \int\limits_{-a/2}^{a/2} \int\limits_{-b/2}^{b/2} xzdxdy \\ -\sigma_0 \int\limits_{-a/2}^{a/2} \int\limits_{-b/2}^{b/2} xydxdy & \sigma_0 \int\limits_{-a/2}^{a/2} \int\limits_{-b/2}^{b/2} (x^2+z^2) dxdy & -\sigma_0 \int\limits_{-a/2}^{a/2} \int\limits_{-b/2}^{b/2} yzdxdy \\ -\sigma_0 \int\limits_{-a/2}^{a/2} \int\limits_{-b/2}^{b/2} xzdxdy & -\sigma_0 \int\limits_{-a/2}^{a/2} \int\limits_{-b/2}^{b/2} yzdxdy & \sigma_0 \int\limits_{-a/2}^{a/2} \int\limits_{-b/2}^{b/2} (x^2+y^2) dxdy \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_0 \int\limits_{-a/2}^{a/2} \int\limits_{-b/2}^{b/2} y^2 dxdy & -\sigma_0 \int\limits_{-a/2}^{a/2} \int\limits_{-b/2}^{b/2} xydxdy & 0 \\ -\sigma_0 \int\limits_{-a/2}^{a/2} \int\limits_{-b/2}^{b/2} xydxdy & \sigma_0 \int\limits_{-a/2}^{a/2} \int\limits_{-b/2}^{b/2} x^2 dxdy & 0 \\ 0 & 0 & \sigma_0 \int\limits_{-a/2}^{a/2} \int\limits_{-b/2}^{b/2} (x^2+y^2) dxdy \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{12}\sigma_0 ab^3 & 0 & 0 \\ 0 & \frac{1}{12}\sigma_0 a^3 b & 0 \\ 0 & 0 & \frac{1}{12}\sigma_0 ab(a^2+b^2) \end{pmatrix} = \frac{1}{12} M_1 \begin{pmatrix} b^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2+b^2 \end{pmatrix}$$

$$I_2 = \begin{pmatrix} \sigma_0 \int\limits_{-b/2}^{b/2} \int\limits_{-c/2}^{c/2} (y^2+z^2) dydz & -\sigma_0 \int\limits_{-b/2}^{b/2} \int\limits_{-c/2}^{c/2} xydydz & -\sigma_0 \int\limits_{-b/2}^{b/2} \int\limits_{-c/2}^{c/2} xzdydz \\ -\sigma_0 \int\limits_{-b/2}^{b/2} \int\limits_{-c/2}^{c/2} xydydz & \sigma_0 \int\limits_{-b/2}^{b/2} \int\limits_{-c/2}^{c/2} (x^2+z^2) dydz & -\sigma_0 \int\limits_{-b/2}^{b/2} \int\limits_{-c/2}^{c/2} yzdydz \\ -\sigma_0 \int\limits_{-b/2}^{b/2} \int\limits_{-c/2}^{c/2} xzdydz & -\sigma_0 \int\limits_{-b/2}^{b/2} \int\limits_{-c/2}^{c/2} yzdydz & \sigma_0 \int\limits_{-b/2}^{b/2} \int\limits_{-c/2}^{c/2} (x^2+y^2) dydz \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_0 \int\limits_{-b/2}^{b/2} \int\limits_{-c/2}^{c/2} (y^2+z^2) dydz & 0 & 0 \\ 0 & \sigma_0 \int\limits_{-b/2}^{b/2} \int\limits_{-c/2}^{c/2} z^2 dydz & -\sigma_0 \int\limits_{-b/2}^{b/2} \int\limits_{-c/2}^{c/2} yzdydz \\ 0 & -\sigma_0 \int\limits_{-b/2}^{b/2} \int\limits_{-c/2}^{c/2} yzdydz & \sigma_0 \int\limits_{-b/2}^{b/2} \int\limits_{-c/2}^{c/2} y^2 dydz \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{12}\sigma_0 bc(b^2+c^2) & 0 & 0 \\ 0 & \frac{1}{12}\sigma_0 bc^3 & 0 \\ 0 & 0 & \frac{1}{12}\sigma_0 b^3 c \end{pmatrix} = \frac{1}{12} M_2 \begin{pmatrix} b^2+c^2 & 0 & 0 \\ 0 & c^2 & 0 \\ 0 & 0 & b^2 \end{pmatrix}$$

$$I_1' = I_1 + M_1 \begin{pmatrix} \frac{b^2}{4} & -\frac{ab}{4} & 0 \\ -\frac{ab}{4} & \frac{a^2}{4} & 0 \\ 0 & 0 & \frac{a^2+b^2}{4} \end{pmatrix} = \frac{1}{12} M_1 \begin{pmatrix} 4b^2 & -3ab & 0 \\ -3ab & 4a^2 & 0 \\ 0 & 0 & 4(a^2+b^2) \end{pmatrix}$$

$$I_2' = I_2 + M_2 \begin{pmatrix} \frac{b^2+c^2}{4} & 0 & 0 \\ 0 & \frac{c^2}{4} & -\frac{bc}{4} \\ 0 & -\frac{bc}{4} & \frac{b^2}{4} \end{pmatrix} = \frac{1}{12} M_2 \begin{pmatrix} 4(b^2+c^2) & 0 & 0 \\ 0 & 4c^2 & -3bc \\ 0 & -3bc & 4b^2 \end{pmatrix}$$

$$I_G = I_1' + I_2' = \frac{1}{12} M_1 \begin{pmatrix} 4b^2 & -3ab & 0 \\ -3ab & 4a^2 & 0 \\ 0 & 0 & 4(a^2+b^2) \end{pmatrix} + \frac{1}{12} M_2 \begin{pmatrix} 4(b^2+c^2) & 0 & 0 \\ 0 & 4c^2 & -3bc \\ 0 & -3bc & 4b^2 \end{pmatrix}$$

$$= \frac{1}{12} \begin{pmatrix} 4b\sigma_0 \left(ab^2+b^2c+c^3\right) & -3a^2b^2\sigma_0 & 0 \\ -3a^2b^2\sigma_0 & 4b\sigma_0 \left(a^3+c^3\right) & -3b^2c^2\sigma_0 \\ 0 & -3b^2c^2\sigma_0 & 4b\sigma_0 \left(a^3+ab^2+b^2c\right) \end{pmatrix}$$

b) $I_G = \frac{1}{12}M \begin{pmatrix} 12L^2 & -3L^2 & 0 \\ -3L^2 & 8L^2 & -3L^2 \\ 0 & -3L^2 & 12L^2 \end{pmatrix}$

Hauptträgheitsmomente=Eigenwerte:

$$0 \stackrel{!}{=} \begin{vmatrix} L^2M-\lambda & -\frac{L^2M}{4} & 0 \\ -\frac{L^2M}{4} & \frac{2L^2M}{3}-\lambda & -\frac{L^2M}{4} \\ 0 & -\frac{L^2M}{4} & L^2M-\lambda \end{vmatrix}$$

$$= (L^2M-\lambda)^2(\tfrac{2L^2M}{3}-\lambda) - 2(-(\tfrac{L^2M}{4}))^2(L^2M-\lambda)$$

$$= (L^2M-\lambda)(-\tfrac{1}{8}L^4M^2 + (L^2M-\lambda)(\tfrac{L^2M}{18}-\tfrac{\lambda}{24}))$$

$$= \tfrac{1}{24}\left(L^2M-\lambda\right)\left(13L^4M^2 - 40L^2M\lambda + 24\lambda^2\right)$$

$$\Rightarrow \lambda_1 = L^2M,\ \lambda_2 = \tfrac{1}{12}L^2M\left(10-\sqrt{22}\right),\ \lambda_3 = \tfrac{1}{12}L^2M\left(10+\sqrt{22}\right)$$

Hauptträgheitsachsen=Eigenvektoren:

$$\text{1. EW:} \left|\begin{array}{ll} 0 - \frac{L^2M}{4}v_y^1 + 0 & = 0 \\ -\frac{L^2M}{4}v_x^1 - \frac{1}{3}L^2Mv_y^1 - \frac{L^2M}{4}v_z^1 & = 0 \\ 0 + -\frac{L^2M}{4}v_y^1 + 0 & = 0 \end{array}\right| \Rightarrow \left|\begin{array}{ll} v_y^1 & = 0 \\ v_x^1 & = -v_z^1 \end{array}\right| \Rightarrow v^1 = \begin{pmatrix} v_0 \\ 0 \\ -v_0 \end{pmatrix}$$

$$\text{2. EW:} \left|\begin{array}{ll} LM^2(1-\frac{1}{12}(10-\sqrt{22}))v_x^2 - \frac{L^2M}{4}v_y^2 & = 0 \\ -\frac{L^2M}{4}v_x^2 + L^2M(\frac{2}{3}-10+\sqrt{22})v_y^2 - \frac{L^2M}{4}v_z^2 & = 0 \\ -\frac{L^2M}{4}v_y^2 + LM^2(1-\frac{1}{12}(10-\sqrt{22}))v_z^2 & = 0 \end{array}\right| \Rightarrow \left|\begin{array}{ll} v_x^2 & = v_z^2 \\ v_y^2 & = \frac{1}{3}(2+\sqrt{22})v_z^2 \end{array}\right| \Rightarrow$$

$$v^2 = \begin{pmatrix} v_0 \\ \frac{1}{3}(2+\sqrt{22})v_0 \\ v_0 \end{pmatrix}$$

$$\text{3. EW:} \left|\begin{array}{ll} LM^2(1-\frac{1}{12}(10+\sqrt{22}))v_x^3 - \frac{L^2M}{4}v_y^2 & = 0 \\ -\frac{L^2M}{4}v_x^3 + L^2M(\frac{2}{3}-10-\sqrt{22})v_y^3 - \frac{L^2M}{4}v_z^3 & = 0 \\ -\frac{L^2M}{4}v_y^3 + LM^2(1-\frac{1}{12}(10+\sqrt{22}))v_z^3 & = 0 \end{array}\right| \Rightarrow \left|\begin{array}{ll} v_x^3 & = v_z^3 \\ v_y^3 & = \frac{1}{3}(2-\sqrt{22})v_z^3 \end{array}\right| \Rightarrow$$

$$v^3 = \begin{pmatrix} v_0 \\ \frac{1}{3}(2-\sqrt{22})v_0 \\ v_0 \end{pmatrix}$$

$\Rightarrow$Einheitsvektoren in Hauptträgheitsachsen-Richtung:

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},\ v_2 = \begin{pmatrix} 1 \\ \frac{1}{3}(2+\sqrt{22}) \\ 1 \end{pmatrix},\ v_3 = \begin{pmatrix} 1 \\ \frac{1}{3}(2-\sqrt{22}) \\ 1 \end{pmatrix}$$

$$T = (\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0) \cdot \left(\frac{1}{12}M \begin{pmatrix} 12L^2 & -3L^2 & 0 \\ -3L^2 & 8L^2 & -3L^2 \\ 0 & -3L^2 & 12L^2 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \right)$$

$$= \frac{3L^2M}{8} + \frac{\frac{1}{3}\sqrt{2}L^2M - \frac{L^2M}{4\sqrt{2}}}{\sqrt{2}} = \frac{7}{12}L^2M$$