Vorbereitung 1 zu E-Dynamik

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$$\begin{vmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{vmatrix} = \overbrace{\delta_{i1}\delta_{j2}\delta_{k3}}^{1,2,3 \to 1 \text{ sonst } 0} + \overbrace{\delta_{i2}\delta_{j3}\delta_{k1}}^{2,3,1 \to 1 \text{ sonst } 0} + \overbrace{\delta_{i3}\delta_{j1}\delta_{k2}}^{3,1,2 \to \text{ sonst } 0} \Rightarrow \text{gerade Anz. Perm.} = 1$$

$$\underbrace{-\delta_{i1}\delta_{j3}\delta_{k2}}_{1,3,2 \to -1 \text{ sonst } 0} \underbrace{-\delta_{i2}\delta_{j1}\delta_{k3}}_{2,1,3 \to -1 \text{ sonst } 0} \underbrace{-\delta_{i3}\delta_{j2}\delta_{k1}}_{3,2,1 \to -1 \text{ sonst } 0} \Rightarrow \text{ungerade Anz. Perm.} = -1$$

$\Rightarrow$ sonst 0 $\Rightarrow \ldots = \varepsilon_{ijk}$

$$\varepsilon_{ijk}\varepsilon_{lmk} = \begin{vmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{vmatrix} \cdot \begin{vmatrix} \delta_{1l} & \delta_{1m} & \delta_{1k} \\ \delta_{2l} & \delta_{2m} & \delta_{2k} \\ \delta_{3l} & \delta_{3m} & \delta_{3k} \end{vmatrix} = \left| (\ldots)\cdot(\ldots) \right|$$

$$= \begin{vmatrix} \delta_{i1}\delta_{1l}+\delta_{i2}\delta_{2l}+\delta_{i3}\delta_{3l} & \delta_{i1}\delta_{1m}+\delta_{i2}\delta_{2m}+\delta_{i3}\delta_{3m} & \delta_{i1}\delta_{1k}+\delta_{i2}\delta_{2k}+\delta_{i3}\delta_{3k} \\ \delta_{j1}\delta_{1l}+\delta_{j2}\delta_{2l}+\delta_{j3}\delta_{3l} & \delta_{j1}\delta_{1m}+\delta_{j2}\delta_{2m}+\delta_{j3}\delta_{3m} & \delta_{j1}\delta_{1k}+\delta_{j2}\delta_{2k}+\delta_{j3}\delta_{3k} \\ \delta_{k1}\delta_{1l}+\delta_{k2}\delta_{2l}+\delta_{k3}\delta_{3l} & \delta_{k1}\delta_{1m}+\delta_{k2}\delta_{2m}+\delta_{k3}\delta_{3m} & \delta_{k1}\delta_{1k}+\delta_{k2}\delta_{2k}+\delta_{k3}\delta_{3k} \end{vmatrix}$$

$$= \begin{vmatrix} \delta_{il}+\delta_{il}+\delta_{il} & \delta_{im}+\delta_{im}+\delta_{im} & \delta_{ik}+\delta_{ik}+\delta_{ik} \\ \delta_{jl}+\delta_{jl}+\delta_{jl} & \delta_{jm}+\delta_{jm}+\delta_{jm} & \delta_{jk}+\delta_{jk}+\delta_{jk} \\ \delta_{kl}+\delta_{kl}+\delta_{kl} & \delta_{km}+\delta_{km}+\delta_{km} & \cancel{\delta_{kk}+\ldots} \end{vmatrix} = \begin{vmatrix} \cancel{3}\delta_{il} & \cancel{3}\delta_{im} & \cancel{3}\delta_{ik} \\ \cancel{3}\delta_{jl} & \cancel{3}\delta_{jm} & \cancel{3}\delta_{jk} \\ \cancel{3}\delta_{kl} & \cancel{3}\delta_{km} & 3 \end{vmatrix}$$

$$= \delta_{kl}\cdot\begin{vmatrix} \delta_{im} & \delta_{ik} \\ \delta_{jm} & \delta_{jk} \end{vmatrix} - \delta_{km}\begin{vmatrix} \delta_{il} & \delta_{ik} \\ \delta_{jl} & \delta_{jk} \end{vmatrix} + 3\begin{vmatrix} \delta_{il} & \delta_{im} \\ \delta_{jl} & \delta_{jm} \end{vmatrix}$$

$$= \begin{vmatrix} \delta_{im} & \delta_{ik}\delta_{kl} \\ \delta_{jm} & \delta_{jk}\delta_{kl} \end{vmatrix} - \begin{vmatrix} \delta_{il} & \delta_{ik}\delta_{km} \\ \delta_{jl} & \delta_{jk}\delta_{km} \end{vmatrix} + 3\left|\ldots\right| = \begin{vmatrix} \delta_{im} & \delta_{il} \\ \delta_{jm} & \delta_{jl} \end{vmatrix} - \begin{vmatrix} \delta_{il} & \delta_{im} \\ \delta_{jl} & \delta_{jm} \end{vmatrix} + 3\left|\ldots\right|$$

$$= (3-2)\begin{vmatrix} \delta_{il} & \delta_{im} \\ \delta_{jl} & \delta_{jm} \end{vmatrix} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$$\varepsilon_{ijk}\varepsilon_{ljk} = \cancel{\delta_{il}\delta_{jj} - \delta_{ij}\delta_{jl} = \delta_{il} - \delta_{il}} \quad \cancel{\sum_k\Big(\sum_j(\varepsilon_{jki}\varepsilon_{jkl})\Big) = \sum_k(\ldots)}$$

$$\sum_k\Big(\sum_j(\varepsilon_{jki}\varepsilon_{jkl})\Big) = \sum_k(\delta_{kk}\delta_{il} - \delta_{kl}\delta_{ik}) = 3\delta_{il} - \delta_{il} = 2\delta_{il}$$

$$\varepsilon_{ijk}\varepsilon_{ijk} = \sum_i\Big(\sum_j\Big(\sum_k(\varepsilon_{ijk}\varepsilon_{ijk})\Big)\Big) = \sum_i\Big(\sum_j(\delta_{ii}\delta_{jj} - \delta_{ij}\delta_{ji})\Big) = \sum_i(3-\delta_{ii}) = 9-3 = 6$$

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$$\vec{a}\times\vec{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} \; ; \; \sum_{i=1}^{3}\sum_{k=1}^{3}\varepsilon_{ijk}a_jb_k\vec{e}_i = \begin{pmatrix} \varepsilon_{123}a_2b_3 + \cancel{\varepsilon_{111}(\ldots)} + \ldots + \varepsilon_{132}a_3b_2 \\ \varepsilon_{231}a_3b_1 + \ldots + \varepsilon_{213}a_1b_3 \\ \varepsilon_{312}a_1b_2 + \ldots + \varepsilon_{321}a_2b_1 \end{pmatrix}$$

$$= \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} \Rightarrow \vec{a}\times\vec{b} = \varepsilon_{ijk}a_jb_k\vec{e}_i$$

$$\vec{\nabla}\times\vec{v} = \begin{pmatrix} \partial_2v_3 - \partial_3v_2 \\ \partial_3v_1 - \partial_1v_3 \\ \partial_1v_2 - \partial_2v_1 \end{pmatrix} \quad ; \; \sum_{i=1}^{3}\sum_{k=1}^{3}\varepsilon_{ijk}\partial_jv_k\vec{e}_i = \overset{(\pi)}{\ldots} = \begin{pmatrix} \partial_2v_3 - \partial_3v_2 \\ \partial_3v_1 - \partial_1v_3 \\ \partial_1v_2 - \partial_2v_1 \end{pmatrix} \Rightarrow ✓$$

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a) $$\vec{\nabla}(\vec{a}\cdot\vec{b}) = \cancel{\vec{\nabla}(a_1b_1+a_2b_2+a_3b_3)} = \cancel{\begin{pmatrix}\partial_1a_ib_i\\\partial_2a_ib_i\\\partial_3a_ib_i\end{pmatrix}} = \begin{pmatrix} b_i\partial_1a_i + a_i\partial_1b_i \\ b_i\partial_2a_i + a_i\partial_2b_i \\ b_i\partial_3a_i + a_i\partial_3b_i \end{pmatrix}$$

$$(\ldots) = \begin{pmatrix} a_i\partial_jb_j \\ a_i\partial_jb_j \\ a_i\partial_jb_j \end{pmatrix} + \begin{pmatrix} b_i\partial_ja_j \\ b_i\partial_ja_j \\ b_i\partial_ja_j \end{pmatrix} + \begin{pmatrix} b_2\partial_1a_2 + b_3\partial_1a_3 - b_2\partial_2a_1 - b_3\partial_3a_1 \\ -b_1\partial_1a_2 + b_1\partial_2a_1 + b_3\partial_2a_3 - b_3\partial_3a_2 \\ -b_1\partial_1a_3 - b_2\partial_2a_3 + b_1\partial_3a_1 - b_2\partial_3a_2 \end{pmatrix} + \begin{pmatrix} a_2\partial_1b_2 + a_3\partial_1b_3 - a_2\partial_2b_1 - a_3\partial_3b_1 \\ -a_1\partial_1b_2 + a_1\partial_2b_1 + a_3\partial_2b_3 - a_3\partial_3b_2 \\ -a_1\partial_1b_3 - a_2\partial_2b_3 + a_1\partial_3b_1 - a_2\partial_3b_2 \end{pmatrix}$$

$$= \begin{pmatrix} a_1\partial_1b_1 + b_1\partial_1a_1 + a_1\partial_2b_2 + b_1\partial_2a_2 + a_1\partial_3b_3 + b_1\partial_3a_3 + b_2\partial_1a_2 + b_3\partial_1a_3 - b_2\partial_2a_1 - b_3\partial_3a_1 + a_2\partial_1b_2 + a_3\partial_1b_3 - a_2\partial_2b_1 - a_3\partial_3b_1 \\ a_2\partial_1b_1 + b_2\partial_1a_1 + a_2\partial_2b_2 + b_2\partial_2a_2 + a_2\partial_3b_3 + b_2\partial_3a_3 - b_1\partial_1a_2 + b_1\partial_2a_1 + b_3\partial_2a_3 - b_3\partial_3a_2 - a_1\partial_1b_2 + a_1\partial_2b_1 + a_3\partial_2b_3 - a_3\partial_3b_2 \\ a_3\partial_1b_1 + b_3\partial_1a_1 + a_3\partial_2b_2 + b_3\partial_2a_2 + a_3\partial_3b_3 + b_3\partial_3a_3 - b_1\partial_1a_3 - b_2\partial_2a_3 + b_1\partial_3a_1 - b_2\partial_3a_2 - a_1\partial_1b_3 - a_2\partial_2b_3 + a_1\partial_3b_1 - a_2\partial_3b_2 \end{pmatrix}$$

$$= \begin{pmatrix} a_i\partial_1b_i \\ a_i\partial_2b_i \\ a_i\partial_3b_i \end{pmatrix} + \begin{pmatrix} b_i\partial_1a_i \\ b_i\partial_2a_i \\ b_i\partial_3a_i \end{pmatrix} \Rightarrow ✓$$

b) $$\vec{\nabla}(\vec{a}\times\vec{b}) = \begin{pmatrix} \partial_2a_1b_2 + \partial_3a_1b_3 - \partial_2b_1a_2 - \partial_3b_1a_3 \\ \partial_1a_2b_1 + \partial_3a_2b_3 - \partial_1b_2a_1 - \partial_3b_2a_3 \\ \partial_1a_3b_1 + \partial_2a_3b_2 - \partial_1b_3a_1 - \partial_2b_3a_2 \end{pmatrix} \; ; \; (\ldots) = \begin{pmatrix} b_1a_i\partial_i \\ b_2a_i\partial_i \\ b_3a_i\partial_i \end{pmatrix} + \begin{pmatrix} a_1b_i\partial_i \\ a_2b_i\partial_i \\ a_3b_i\partial_i \end{pmatrix} + (\ldots a \leftrightarrow b)$$

$$+(\ldots a \leftrightarrow b)$$

$$\Rightarrow \begin{pmatrix} \partial_2b_2a_1 + \partial_3b_3a_1 - \partial_2a_2b_1 - \partial_3a_3b_1 + (*) \\ \partial_1b_1a_2 + \partial_3b_3a_2 - \partial_1a_1b_2 - \partial_3a_3b_2 + (**) \\ \partial_1b_1a_3 + \partial_2b_2a_3 - \partial_2a_2b_3 - \partial_1a_1b_3 + (***) \end{pmatrix}$$

$= (\ldots)$ $\Rightarrow$ ✓ $(+\ldots a \leftrightarrow b)$

$(*) = \partial_1 a_1 b_1 - \partial_1 b_1 a_1$; $(**) = \partial_2 a_2 b_2 - \partial_2 b_2 a_2$; $(***) = \partial_3 a_3 b_3 - \partial_3 b_3 a_3$

b) $\vec{\nabla}(\vec{a} \times \vec{b}) = \partial_1(a_2 b_3 - a_3 b_2) + \partial_2(a_3 b_1 - a_1 b_3) + \partial_3(a_1 b_2 - a_2 b_1) + (\ldots a \leftrightarrow b)$

$(\ldots) = b_1(\partial_2 a_3 - \partial_3 a_2) + b_2(\partial_3 a_1 - \partial_1 a_3) + b_3(\partial_1 a_2 - \partial_2 a_1)$

$- a_1(\partial_2 b_3 - \partial_3 b_2) + a_2(\partial_3 b_1 - \partial_1 b_3) - a_3(\partial_1 b_2 - \partial_2 b_1)$

$= \partial_1(-a_3 b_2 + a_2 b_3 + a_2 b_3 - a_3 b_2) + \partial_2(a_3 b_1 + a_3 b_1 - a_1 b_3 - a_1 b_3) + \partial_3(a_1 b_2 + a_1 b_2 - a_2 b_1 - a_2 b_1)$

$= (\cdot\cdot) \Rightarrow$ ✓