Übung 1 zu Elektrodynamik   Julian Bergmann

Nr. 4

a) $$\int_\alpha^\beta f(x)\delta'(x-x_0)\,dx = [f(x)\delta(x-x_0)]_\alpha^\beta - \int_\alpha^\beta f'(x)\delta(x-x_0)\,dx$$
$$= \underbrace{f(\beta)\delta(\beta-x_0)}_{0} - \underbrace{f(\alpha)\delta(\alpha-x_0)}_{0} - f'(x_0) = -f'(x_0)$$

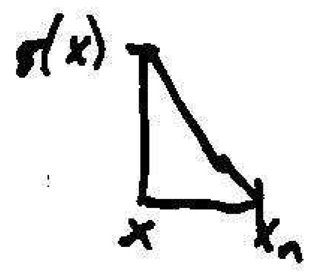
$$\int_\alpha^\beta f(x)\delta^{(n)}(x-x_0)\,dx = [f(x)\delta^{(n-1)}(x-x_0)]_\alpha^\beta - \int_\alpha^\beta f^{(1)}(x)\delta^{(n-1)}(x-x_0)\,dx$$

$\delta(\alpha-x_0)$ mit $\alpha<x_0$ ist 0. $\delta(\alpha+\varepsilon-x_0)=0$, mit $\forall\varepsilon$ klein genug, dass $\alpha+\varepsilon<x_0$. Ebenso $\delta(\beta-x_0)=0$, $\delta(\beta-\varepsilon-x_0)=0$, $\forall\varepsilon: \beta>\beta-\varepsilon>x_0$.

Also $\forall n\in\mathbb{N}: \delta^{(n)}(\alpha-x_0)=0$ mit $\alpha\neq x_0$.

Also $$\int_\alpha^\beta f(x)\delta^{(n)}(x-x_0)\,dx = -\int_\alpha^\beta f^{(1)}(x)\delta^{(n-1)}(x-x_0)\,dx = \int_\alpha^\beta f^{(2)}(x)\delta^{(n-2)}(x-x_0)\,dx$$
$$= \dots = (-1)^{n-1}\int_\alpha^\beta f^{(n-1)}(x)\delta^{(1)}(x-x_0)\,dx = (-1)^n f^{(n)}(x_0)$$

b) $$\int_\alpha^\beta \delta(g(x))f(x)\,dx = \sum_n^{\alpha<x_n<\beta}\int_{x_n-\varepsilon}^{x_n+\varepsilon}\delta(g(x))f(x)\,dx = \sum_n^{\alpha<x_n<\beta}\int_{x_n-\varepsilon}^{x_n+\varepsilon}\delta\left(g(x)\frac{x-x_n}{x-x_n}\right)f(x)\,dx$$
$$= \sum_n^{\alpha<x_n<\beta}\int_{x_n-\varepsilon}^{x_n+\varepsilon}\delta(g'(x_n)(x-x_n))f(x)\,dx$$



$$\lim_{\varepsilon\to 0^+}\sum_n^{\alpha<x_n<\beta}\int_{x_n-\varepsilon}^{x_n+\varepsilon}\delta(g'(x_n)(x-x_n))f(x)\,dx = \int_\alpha^\beta\sum_n^{\alpha<x_n<\beta}\delta(g'(x_n)(x-x_n))f(x)\,dx$$

Wenn $g'(x)>0$: Substitution: $u=g'(x_n)x$, $dx=\frac{1}{g'(x_n)}du$

$$\dots = \int_{\alpha g'(x_n)}^{\beta g'(x_n)}\sum_n^{\alpha g'(x_n)<u_n<\beta g'(x_n)}\delta\left(\frac{g'(x_n)}{g'(x_n)}u - \underbrace{g'(x_n)x_n}_{u_n}\right)f\left(\frac{u}{g'(x_n)}\right)\frac{1}{g'(x_n)}\,du$$
$$= \sum_n^{\alpha g'(x_n)<u_n<\beta g'(x_n)} f\left(\frac{u_n}{g'(x_n)}\right)\cdot\frac{1}{g'(x_n)} = \sum_n^{\alpha<x_n<\beta} f(x_n)\cdot\frac{1}{g'(x_n)} = \int_\alpha^\beta\sum_n^{\alpha<x_n<\beta}\frac{1}{g'(x_n)}\delta(x-x_n)f(x)\,dx$$
$$\Rightarrow \int_\alpha^\beta\delta(g(x))f(x)\,dx = \int_\alpha^\beta\sum_n^{\alpha<x_n<\beta}\frac{1}{g'(x_n)}\delta(x-x_n)f(x)\,dx$$
$$\Rightarrow \delta(g(x)) = \sum_n^{\alpha<x_n<\beta}\frac{1}{g'(x_n)}\delta(x-x_n)$$

Wenn $g'(x)<0$:
$$\dots = \sum_n\int_{-\alpha|g'(x)|}^{-\beta|g'(x_n)|}\frac{1}{|g'(x_n)|}\delta(u-u_n)f\left(\frac{u}{g'(x_n)}\right)du$$
$$= \sum_n^{-\alpha|g'(x_n)|>u_n>-\beta|g'(x_n)|}\int_{-\alpha|g'(x_n)|}^{-\beta|g'(x_n)|}\frac{1}{|g'(x_n)|}\delta(u-u_n)f\left(\frac{u}{g'(x_n)}\right)du = \sum_n^{-\alpha|g'(x_n)|>u_n>-\beta|g'(x_n)|}\frac{1}{|g'(x_n)|}f\left(\frac{u_n}{g'(x_n)}\right)$$
$$= \sum_n^{\alpha<x_n<\beta}\frac{1}{|g'(x_n)|}f(x_n) = \int_\alpha^\beta\sum_n^{\alpha<x_n<\beta}\frac{1}{|g'(x_n)|}\delta(x-x_n)f(x)\,dx$$
$$\Rightarrow\int_\alpha^\beta\delta(g(x))f(x)\,dx = \int_\alpha^\beta\sum_n^{\alpha<x_n<\beta}\frac{1}{|g'(x_n)|}\delta(x-x_n)f(x)\,dx$$
$$\Rightarrow \delta(g(x)) = \sum_n^{\alpha<x_n<\beta}\frac{1}{|g'(x_n)|}\delta(x-x_n).$$

Da bei $g'(x_n)>0: g'(x_n)=|g'(x_n)|$ ist $\delta(g(x)) = \sum_n^{\alpha<x_n<\beta}\frac{1}{|g'(x_n)|}\delta(x-x_n)$ in beiden Fällen gültig.