Theorie der höheren Mechanik

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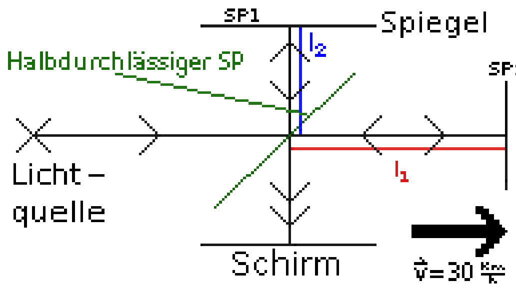
1 Spezielle Relativitätstheorie

$F = m\ddot{\vec{r}}$ invariant unter Galilei-Transformation:
$\vec{r}' = \vec{r} + \vec{V} \cdot t$ (V const). $\Rightarrow \dot{\vec{r}}' = \vec{v}' = \dot{\vec{r}} + \vec{V} = \vec{v} + \vec{V}$
$\Rightarrow F = m\ddot{\vec{r}}'$ (Galilei-Invarianz)

1.1 Michelson-Experiment

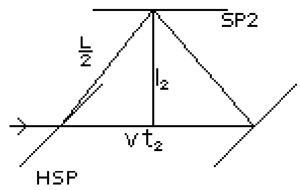


Laufzeit auf Weg 1:

$t_1 = \frac{l_1}{c-v} + \frac{l_1}{c+v} = \frac{2l_1c}{c^2-v^2} = 2\frac{l_1}{c}\frac{1}{1-\frac{v^2}{c^2}}$ ($c \pm v$: Galilei)
$\beta \equiv \frac{v}{c}$; $\quad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$
$\Rightarrow t_1 = 2\frac{l_1}{c}\gamma^2$

Laufzeit auf Weg 2:

$L = 2\sqrt{(\frac{vt_2}{2})^2 + l_2^2}; 4l_2^2 + (vt_2)^2 = (ct_2)^2$
$\Rightarrow t_2^2 = 4\frac{l_2^2}{c^2}\gamma^2 \Rightarrow t2 = 2\frac{l_2}{c}\gamma$



$\Delta t = t_2 - t_1 = \frac{2}{c}(l_2\gamma - l_1\gamma^2)$; $\quad \Delta t' = \frac{2}{c}(l_1\gamma - l_2\gamma^2)$
$\Delta t \to \Delta t'$: keine Änderung auf dem Schirm!
($P + A \to x + \pi^0$ (π^0 bewegt sich ca. mit 0,95c und strahlt 2γ ab),
Genauigkeit $10^{-4} \Rightarrow$ c konstant in allen Bezugsystemen)

1.2 Einstein'sche Postulate

1. Lichtgeschwindigkeit hat zahlenmäßig gleichen Wert in allen Bezugsystemen.
2. Naturgesetze sind in allen Intertialsystemen gleich.

1.3 Konsequenzen

Gegeben:

Ein Stock (z.b. 1m), an dessen einem Ende eine Lichtquelle und ein Schirm, am anderen Ende ein Spiegel ist. Die Lichtquelle sendet in regelmäßigen Zeitintervallen Lichtimpulse, die über den Schirm „getickt“ werden.
Ein anderer Beobachter mit der gleichen Ausrüstung befindet sich im System S', das sich im Gegensatz zu ihrem System S in z-Richtung bewegt.

Die Zeitdauer zwischen 2 Ticks:
Der Beobachter in S': $t = 2\frac{l}{c}\gamma$ (Zeitdauer zwischen 2 Ticks auf bewegter Uhr in S)
$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

Zurückgelegte Strecke in den Inertialsystemen:

	Emission	Absorption
S'	$t' = 0,\ x' = 0,\ y' = 0,\ z' = 0$	$t' = 2\frac{l}{c},\ x' = 0,\ y' = 0,\ z' = 0$
S	$t = 0,\ x = 0,\ y = 0,\ z = 0$	$t = 2\frac{l}{c}\gamma,\ x = 0,\ y = 0,\ z = vt = 2l\beta\gamma$

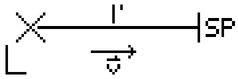
$S': \; c^2\Delta t^2 - (\Delta s)^2 = c^2(2\frac{l}{c})^2 - 0) = 4l^2$
$S: \; c^2(2\frac{l}{c}\gamma)^2 - (2l\beta\gamma)^2 = 4l(\underbrace{\gamma^2 - \beta^2\gamma^2}_{1})$
$\Rightarrow \; c^2\Delta t^2 - (\Delta s)^2$ invariant!

1.4 Beispiel

2 Ereignisse in S am gleichen Ort: $\Delta t = 3s$; $\Delta t'$ *von* S' *aus* $= 5s$
$9s^2 \cdot c^2 \stackrel{!}{=} 25s^2 \cdot c^2 - (\Delta s)^2 \Rightarrow (\Delta s)^2 = 16s^2 \cdot c^2 \Rightarrow \Delta s = 4s \cdot c \approx 1,2 \cdot 10^6 km$
Aus dem anderen Bezugsystem finden beide Ereignisse an vollkommen unterschiedlichen Orten statt!

1.5 Zeitdiletation und Längenkontraktion

Eigenzeit: gemessene Zeit einer Uhr, die in System des Ereignisses ruht.



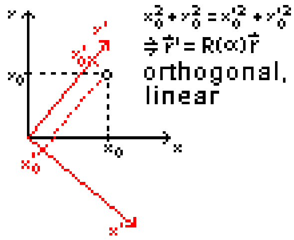
Lichtuhr ruhe in S', parallel zu $\vec{v}$, Länge l' (Eigenlänge) in S' gemessen.
$\Delta t' = \frac{2l'}{c} \Rightarrow \Delta t = \frac{2l'}{c}\gamma = \gamma\Delta t'$ (Zeitdiletation)

$$\left.\begin{array}{ll} \text{Hinweg:} & c\Delta t_1 = l' + v\Delta t_1 \\ \text{Rückweg:} & c\Delta t_1 = l' - v\Delta t_1 \end{array}\right\} \Delta t = \Delta t_1 + \Delta t_2 = \frac{l'}{c-v} + \frac{l'}{c+v}$$

$= 2\frac{l'}{c} \cdot \frac{1}{1-(\frac{v}{c})^2} = 2\frac{l}{c}\gamma^2 \neq \gamma\Delta t' \Rightarrow$ andere Länge!
$l\gamma = l' \Rightarrow l = \frac{1}{\gamma}l' = \sqrt{1-\beta^2}l'$ (Längenkontraktion)

1.6 Lorentztransformation

Abstände und Beträge von Vektoren bleiben bei Transformationen der Koordinatensysteme gleich.
Bei griechischen Indizes wird ab 0 gezählt, bei lateinischen ab 1.
Vierervektor: $x^\mu = (ct, \vec{x}) = (ct, x, y, z)$ „Kontravarianter Vierervektor“



Gesucht: Transfo.: $x^\mu \to x'^\mu$
1. Einstein'sche Äquivalenz $\Rightarrow$ Transfo. linear! $\Rightarrow$ $x' = Lx$ (L: 4×4-Matrix)
2. $c^2t^2 = x^2 + y^2 + z^2 \Leftrightarrow c^2t'^2 = x'^2 + y'^2 + z'^2 \Rightarrow c^2t^2 - \vec{x}^2$ invariant unter L!
Def. kovarianter Vierervektor: $x_\mu = (ct, -\vec{x}) \Rightarrow$ Skalarprodukt: $x_\mu x^\mu = c^2t^2 - \vec{x}^2$
(Pseudo-euklidische Metrik, Invariante)

Rehenfolge und Position von Indizes oben und unten führen zu anderen Matrizen/Elementen und sind zu beachten!
Wenn in einem Term ein Index mehrfach vorkommt, wird über diesen Index mit allen möglichen Indizes summiert.

$$x_\mu = g_{\mu\nu}x^\nu \Rightarrow g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \text{(metrischer Tensor)}$$

$x^\mu = g^{-1}_{\mu\nu}x_\nu = g_{\mu\nu}x_\nu \Rightarrow x^\mu = g^{\mu\nu}x_\nu = g_{\mu\nu}x_\nu$

Wichtig: Bei Kontraktionen (Skalarprodukten) 1 Index unten & 1 Index oben.

$g^{\mu\varrho}g_{\varrho\nu} = \mathbb{1} = \delta^\mu{}_\nu$ (4D-Kronecker-Delta)

$Lorentz-Transfo.: x'^\mu = L^\mu{}_\nu x^\nu$; gesucht: $L^\mu{}_\nu = L^{\mu\varrho}g_{\varrho\nu}$ mit $x'^\mu x'_\mu = x^\mu x_\mu$

Bewegung der Bezugssysteme entlang der z-Richtung!

$x'^\mu = L^\mu{}_\nu x^\nu$; $L = L(\nu)$; $L(0) = \mathbb{1}$

Bewegung in z-Richtung: $x' = x,\ y' = y \Rightarrow x^{1'} = x^1,\ x^{2'} = x^2$

$$L = (L^\mu{}_\nu) = \begin{pmatrix} L^0{}_0 & 0 & 0 & L^0{}_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ L^3{}_0 & 0 & 0 & L^3{}_3 \end{pmatrix}; \ x'_\mu x'^\mu \stackrel{!}{=} x_\mu x^\mu$$

$$x'_\mu x'^\mu = \underbrace{L_\mu{}^\nu x_\nu}_{x'_\mu} L^\mu{}_\varrho x^\varrho = \underbrace{L_\mu{}^\nu L^\mu{}_\varrho}_{\cong\delta} x_\nu x^\varrho = (L^T)^\nu{}_\mu L^\mu{}_\varrho x_\mu x_\varrho$$

$= (L^T L)^\nu{}_\varrho x_\nu x^\varrho = x_\lambda x^\lambda$

$(L^T L)^\nu{}_\varrho = \delta^\nu{}_\varrho \Rightarrow$ orthogonal $\Rightarrow\ (L^T L)^\nu{}_\varrho = \mathbb{1}$

$det(L^T L) = +1 = det(L^T)det(L) = 1$

$det(L) = +1$: eigentliche Lorentztransformation

$\delta^0{}_0 = 1 = L^{T0}{}_\varrho L^\varrho{}_0 = L^0{}_0 L^0{}_0 + L^{T0}{}_3 L^3{}_0 = (L^0{}_0)^2 - (L^3{}_0)^2$

$$L^T = (L^T)^\mu{}_\nu = L_\nu{}^\mu = g_{\nu\alpha}L^\alpha{}_\beta g^{\beta\mu} = \begin{pmatrix} L^0{}_0 & 0 & 0 & -L^0{}_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -L^3{}_0 & 0 & 0 & L^3{}_3 \end{pmatrix}$$

$\delta^3{}_3 = 1 = -(L^0{}_3)^2 + (L^3{}_3)^2$

$\delta^0{}_3 = 0 = L^0{}_0 L^0{}_3 - L^3{}_0 L^3{}_3$

$det(L) = +1 \Rightarrow L^0{}_0 L^3{}_3 - L^3{}_0 L^0{}_3$

$L^0{}_0 = \sqrt{1 + (L^0{}_3)^2};\ L^3{}_0 = L^0{}_3;\ L^3{}_3 = L^0{}_0$

$t = t' = 0 wenn z' = z = 0:\ z = vt$ (Nullpunkt von K$'$ in K) $= x^3 = \beta x^0,\ z' = 0 = x'^3$

Allgemein: $x'^3 = L^3{}_\nu x^\nu = L^3{}_0 x^0 + L^3{}_3 x^3 = (L^3{}_0 + \beta L^3{}_3)x^0 = 0$

$\Rightarrow\ L^3{}_0 = -\beta L^3{}_3$ eingesetzt in $L^3{}_3 = \sqrt{1 + (L^0{}_3)^2} = \sqrt{1 + (L^3{}_3)^2} = \gamma$

$$\Rightarrow L^\mu{}_\nu = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}$$

$x' = Lx,\ x = L^{-1}x' = L^T x';\ L^{-1} = L$ mit $-\beta\gamma \to \beta\gamma$

Transo. $\underset{v \to 0}{\longrightarrow}$ Galilei, Transfo. $\underset{v \to c}{\longrightarrow}$ *Grenzgeschw.*

$x' = x,\ y' = y,\ z' = \gamma(z - vt),\ t' = \gamma(t - \frac{\beta}{c}z)$

1.7 Beispiele

1. Stab der Länge l entlang z-Achse ruhend in K. Länge in K′=? $\Delta z = l$
   Naiv: $\Delta z' = \gamma \Delta z$ ↯
   $z_1 - z_2 = l = (z_1' - z_2')\gamma \Rightarrow \Delta z = \Delta z' \gamma$
2. Ruhende Uhr in K sei z_1 mit t_1 von K′ aus $T_1' = \gamma(t_1 - \frac{\beta}{c} z_1), T_2' = \gamma(t_2 - \frac{\beta}{c} z_1)$,
   $\Delta t' = \gamma \Delta t > \Delta t$ (bewegte Uhr langsamer)

1.8 Experiment

μ^- „schweres Elektron“, $\mu^- \Rightarrow e^- + \nu + \nu, P + X \to \pi + x^*, \pi \to \mu + \nu, T_{\frac{1}{2}} = 2,2 \cdot 10^{-6}$
Weg des μ (nr): $s \approxeq c * T_{\frac{1}{2}} = 3 \cdot 10^5 \frac{km}{s} * 2,2 \cdot 10^{-6} s \approx 0,7 km$
$T'_{\frac{1}{2}} = \gamma \cdot 2,2 \cdot 10^{-6} s \Rightarrow \approx$10 tut es!

1.9 Additionstheorem der Geschwindigkeiten

K′ bewegt sich mit v gegenüber K. Teilchen in K′ mit w′. w in K=?
$w' = \frac{dz'}{dt'},\ w = \frac{dz}{dt}$
Lorentz: $z' = \gamma(t - \frac{\beta}{c} z)$
$\Rightarrow dz' = \gamma(dz - wdt),\ dt' = \gamma(dt - \frac{\beta}{c} dz),\ dz = wdt$
$\Rightarrow w' = \frac{dz'}{dt'} = \frac{w - v}{1 - \frac{\beta}{c} w}$
$\Rightarrow w = \frac{v + w'}{1 + \frac{vw'}{c^2}}\ w \underset{\beta \to 0}{\longrightarrow} v + w',\ w \underset{w = c}{\longrightarrow} \frac{v + c}{1 + \beta} = c \frac{1 + \beta}{1 + \beta} = c$ (obere Grenze!)

1.10 Artigket

3 Ergebnisse betrachten: 1. $z = 0$, $t = 0$. sonst noch 2. und 3. (siehe Zeichnung)
2 mit Lichtimpuls erreichbar, 2′ nicht.
Zeitartig: $\Delta s^2 = c^2 \Delta t^2 - (\Delta z)^2 > 0 \Rightarrow \exists$ System mit $\Delta z = 0$
. Lichtartig: $\Delta s^2 = 0$ für $\Delta z = c\Delta t$; liegen auf dem Lichtkegel.
Raumartig: $\Delta s^2 = c^2 \Delta t^2 - (\Delta z)^2 < 0 \Rightarrow \exists$ System mit $\Delta t = 0$.
Artigeit ist Lorentz-invariant! (Wird durch Transformation nicht geändert)
Nur gleichartige Ereignisse können kausal verbunden werden.

1.11 Lorentztensor und Rechenregeln

a^μ Quadrupel mit $a^\mu = L^\mu{}_\nu a^\nu \to a^\mu$ Vierervektor.
b^μ Vierervektor $\to ab = a^\mu b_\mu = a_0 b_0 - \vec{a}\vec{b}$
$a^\mu b^\nu$ (Türiadisches Produkt)
$\to a'^\mu b'^\nu = L^\mu{}_\varrho a^\varrho L^\nu{}_\lambda b^\lambda = L^\mu{}_\varrho a^\varrho b^\lambda L^\nu{}_\lambda = L^\mu{}_\varrho a^\varrho b^\lambda (L^T)_\lambda{}^\nu = (LabL^T)^{\mu\nu}$

$T'^{\mu\nu} = L^\nu{}_\varrho T^{\varrho\lambda} (L^T)_\lambda{}^\nu = (LTL^T)^{\mu\nu}$, dann nennt man T Lorentztensor.

2 Relativistische Mechanik

2.1 Grundlegende Begriffe

Äquivalenzprinzip $\rightarrow$ Grundgleichungen (Bewegungsgleichungen) müssen gleich sein (kovariant).
Kovarianz $\rightarrow$ beide Seiten der Gleichung muss sich gleichartig tranformieren können.
Skalar $\rightarrow$ Invariant (Betragsgleich transformiert).
$a^\mu = c$ (n. kovariant), $a^\mu b\mu = c$ (kovariant), $a^\mu = b^\mu$ (kovariant), $T^{\mu\nu} = b^\mu$ (n. kovariant), $T^{\mu\nu} a_\nu = b^\mu$ (kovariant)

$x^\mu(t)$; Eigenzeit: in Ruhesystem : $(\Delta s)^2 = c^2(\Delta t)^2 = c^2(\Delta t')^2 - (\Delta x')^2$ L-invariant.
$\Rightarrow \Delta t$ Lorentzinvariant $\Delta t = \Delta\tau$ (Eigenzeit).
$\left(= (\Delta t')^2(c^2 - (\frac{\Delta x'}{\Delta t'})^2) = c^2(\Delta t')^2(1 - \frac{v^2}{c2}) = c^2(\Delta t')^2 \frac{1}{\gamma^2}\right)$

2.2 Definition

a) Vierergeschwindigkeit: $u^\mu = \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt}\frac{dt}{d\tau} = (c, \vec{v})\gamma$

b) Viererbeschleunigung: $a^\mu = \frac{du^\mu}{d\tau} = \frac{d^2x^\mu}{d\tau^2} = \frac{du^\mu}{dt}\frac{dt}{d\tau} = \gamma\frac{du^\mu}{dt} = \gamma\frac{d}{dt}(\gamma(t)(c, v(t)))$

c) Viererimpuls: $p^\mu = m_0 u^\mu$ (Ruhemasse m_0 Skalar)
$p^2 = p^\mu p_\mu = m_0^2 u^\mu u_\mu = m_0^2(u^0u_0 - \vec{u}^2) = m_0^2(\gamma^2c^2 - \gamma^2v^2)$
$= m_0\gamma(c^2 - v^2) = c^2m_0^2\gamma^2(1 - \frac{v^2}{c^2}) = m_0^2c^2$ Zeitartig!

2.3 Bewegungsgleichung

$\frac{d}{d\tau}p^\mu = F^\mu \overset{?}{\Leftrightarrow} \frac{dp_i^{nr}}{dt} = F_i^{nr}$
$\gamma\frac{dp^\mu}{dt} = F^\mu \Rightarrow \frac{dp^\mu}{dt} = \frac{1}{\gamma}F^\mu$
$$\underbrace{\frac{1}{\gamma}F^i}_{Minkowski} = \underbrace{F_i^{nr}}_{Newton}$$
$\frac{dp^i}{dt} = F_i^{nr}$ $(i = 1, 2, 3) \Rightarrow F^\mu = (F^0, \gamma\vec{F}^{nr})$
$\vec{F}^{nr} = \frac{d}{dt}(\gamma\vec{p}) = \frac{d}{dt}(\gamma m_0\vec{v}) = \frac{d}{dt}(m\vec{v})$ (m_0: geschwindigkeitsunabhängige Masse)
$m = m_0\gamma = m_0\frac{1}{\sqrt{1-\beta^2}}$
$F^0 = \frac{d}{dt}p^0 = \gamma\frac{d}{dt}(\gamma m_0 c) = \gamma\frac{d}{dt}(mc)$ nach Def!
$F^\mu = \frac{dp^\mu}{d\tau} \quad || \quad u_\mu\sum_\mu \Rightarrow u_\mu F^\mu = u_\mu\frac{dp^\mu}{d\tau} = u_\mu\frac{d}{d\tau}(m_0u^\mu) = m_0(\frac{d}{d\tau}u^\mu)u_\mu = m_0\frac{d}{2d\tau}(u^\mu u_\mu)$
$= m_0\frac{d}{d\tau}u^2$
$\frac{d}{d\tau}u^2 = \frac{d}{d\tau}(u^\mu u_\mu) = \frac{d}{d\tau}c^2 = 0$
$\Rightarrow u^\mu F^\mu = 0 = u_0F^0 + u_iF^i \Rightarrow F^0 = -\frac{1}{u_0}(u_iF^i)$
$F^0 = \frac{1}{c}\gamma\vec{k}\vec{v}$ (Leistung)
$\vec{F} = \gamma\frac{d}{dt}(m\vec{v}) = \gamma(m\frac{dv}{dt} + v\frac{dm}{dt})$, $(m = \gamma(t)m_0)$

Kovariant Definition: $p^\mu = m_0u^\mu$, $u^\mu = \frac{dx^\mu}{d\tau} = \gamma\frac{dx^\mu}{dt} = \gamma(c, \vec{v})$
$\Rightarrow p^\mu = \underbrace{\gamma m_0}_{m}(c, \vec{v}) = m(c, \vec{v})$; $p^2 = p^\mu p_\mu = m_0^2c^2$

Vierkraft: $F^\mu = \frac{dp^\mu}{d\tau} = \gamma \frac{dp^\mu}{dt}$ (Minkowski-Kraft)
$\frac{1}{y} F^\mu = \frac{dp^\mu}{dt}$ (kovariant, nicht mehr manifest kovariant, da nicht offensichtlich kovariant)
$i = 1,2,3.\ K_i^{kl} = \frac{1}{y} F_i = \frac{dp^i}{dt} = \frac{d}{dt}(\gamma m_0 \vec{v}) \underset{\gamma \to 1}{\longrightarrow} \frac{d}{dt}(m_0 v_i)$
$F^0 = \frac{dp^0}{d\tau} = \gamma \frac{dp^0}{dt} = \gamma \frac{d}{dt}(\gamma(t) m_0 c) = \gamma m_0 c \frac{d}{dt} \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{c} \frac{1}{(1-\frac{v^2}{(}c^2))^2} \frac{d}{dt} \frac{m_0}{2} \vec{v}^2$
$F^0 = \frac{1}{c} \gamma \vec{k} \vec{v};\ k = -\vec{\nabla} v \to \vec{k} \vec{v} = -\vec{\nabla} \vec{v} \vec{v} = \frac{dv}{dt}$
$F = \frac{1}{c} \gamma \vec{k} v = \gamma \frac{d}{dt}(\gamma m_0 c) = -\frac{dv}{dt} \frac{1}{c} \gamma \Rightarrow \frac{d}{dt}(\gamma m_0 c^2 + v) = 0$

2.4 Energie und Impuls

Energieerhaltung mit $T = \gamma m_0 c^2 = \frac{m_0 c^2}{\sqrt{1-\frac{x^2}{c^2}}}$
$T = \frac{m_0 c^2}{\sqrt{1-\frac{x^2}{c^2}}} = m_0 c^2 (1 - \frac{v^2}{c^2}^{\frac{1}{2}}) \cong m_0 c^2 (1 + \frac{v^2}{2c^2}) = m_c^2 + \frac{1}{2} m_0 v^2$
$\tilde{T} = T - m_0 c^2 \Rightarrow p^\mu = (\frac{1}{c} T, \gamma m = \vec{v})$
$p^2 = m_0^2 c^2 = \frac{1}{c^2} T^2 - (\gamma m_0 \vec{v})^2 \Rightarrow T^2 = p^2 c^2 + \gamma^2 m_0^2 v^2 c^2 = m_0^2 c^4 + m^2 v^4 c^2$
$\Rightarrow T = \sqrt{\vec{p}^2 c + m_0^2 c^4}$ $(T_{nr} = \frac{p}{2m})$.
bel $m_0, \vec{p} \to \infty$: $T \to pc$.
$m_0 = 0$, $T = |\vec{p}| c$ nur möglich für $v = c$!
$\vec{F} = \frac{d\vec{p}}{dt},\ \vec{F} = 0 \Rightarrow \frac{d\vec{p}}{dt} = 0 \Rightarrow \vec{p}$ const. und $p^0 = \gamma m_0 c = \frac{T}{c}$

2.5 Dopplereffekt für Licht

$E = pc = \hbar\omega,\ \hbar = \frac{h}{2\pi},\ \omega = 2\pi$
Vierervektor des Photons: $p^\mu = (\frac{1}{c} E \gamma \vec{p}) = (\frac{\hbar\omega}{c}, \gamma\vec{p})$ im Ruhesystem der Lichquelle.
$p_x' = p_x,\ p_y' = p_y,\ p_z' = \gamma(p_z - \beta \frac{\hbar\omega}{c}),\ p_0' = \gamma(-\beta p_z + p_0)$
$E' = \gamma(E - v p_z) \Rightarrow \omega' = \gamma(\omega - \frac{v}{\hbar} p_z)$

1) Photon bewegt sind in z-Richtung: $p_z = \frac{E}{c} \Rightarrow \omega' = \gamma(\omega - \beta\omega) = \gamma\omega(1-\beta)$
2) Photon unter Winkel φ zur z-Achse: $p_z = |\vec{p}| cos(\varphi) \Rightarrow \omega' = \omega\gamma(1 - \beta cos(\varphi))$ (sol-
$\varphi = \frac{\pi}{2} \Rightarrow \omega' = \gamma\omega = \frac{\gamma N}{t}$ (transversaler Dopplereffekt).

che Winkeländerung zur Projektion auch Abaration genannt.)

3 Systeme von Teilchen

3.1 Innere und äußere Kraft/Impulserhaltung

$m_k \ddot{\vec{r}}_k = \vec{F}_k^e + \vec{F}_k^i$ (k=1,...,n) ... $= \frac{d}{dt} \vec{P}_k$
(innere Kraft: zwischen Teilchensystem; äußere Kraft: von außen auf das Teilchensystem)
$\vec{F}_k^i = \sum_{k=1}^N \vec{F}_{lk}^i$
$\sum_{k=1}^N \frac{d\vec{p}_k}{dt} = \sum_k (\vec{F}_k^e + \vec{F}_k^i)$
$\vec{P} = \sum_k \vec{p}_k;\ \vec{F}^e = \sum_k \vec{F}_k^e,\ \vec{F}^i = \sum_k \vec{F}_k^i$
$\frac{d}{dt} \vec{P} = \vec{F}^e + \vec{F}^i$

$\frac{d}{dt}\vec{P} = 0 \Rightarrow \vec{F}^e + \vec{F}^i = 0 \Rightarrow \ P\ constant$

Newton: $\vec{F}^i_{lk} = -\vec{F}^i_{kl}$
$\vec{F}^i = \sum\limits_k \vec{F}^i_k = \sum\limits_k \sum\limits_{\substack{l \\ l \neq k}} \vec{F}^i_{lk} = 0$
$\Rightarrow \frac{dP}{dt} = \vec{F}^e \Rightarrow P$ erhalten wenn $\vec{F}^e = 0$

3.2 Drehimpulserhaltung

$\vec{l} = \vec{r} \times \vec{p} \Rightarrow \frac{d\vec{l}}{dt} = \dot{\vec{r}} \times \vec{p} + \vec{r} \times \dot{\vec{p}} = \vec{r} \times \dot{\vec{p}} = \vec{m} \Rightarrow lkonstant$
$\vec{l}_k = \vec{r}_k \times \vec{p}_k,\ \vec{L} = \sum\limits_k \vec{l}_k$
$\frac{dL}{dt} = \sum\limits_k \frac{d\vec{l}_k}{dt} = \sum\limits_k (\vec{r}_k \times \dot{\vec{p}}) = \sum\limits_k (\vec{r}_k \times \vec{F}_k) = \sum\limits_k (\vec{r}_k \times (\vec{F}^i_k + \vec{F}^e_k))$
$= \sum\limits_k (\vec{r}_k \times \vec{F}^e_k) + \sum\limits_k (\vec{r}_k \times \sum\limits_l \vec{F}^i_{lk})$
$\sum\limits_k \sum\limits_l (\vec{r}_k \times \vec{F}^i_{lk}) = \frac{1}{2} \sum\limits_{lk} (\vec{r}_k \times \vec{F}^i_{lk} + \vec{r}_l \times \vec{F}^i_{kl}) = \frac{1}{2} \sum\limits_{lk} ((\vec{r}_k - \vec{r}_l) \times \vec{F}^i_{lk})$
$= 0$ wenn $\vec{F}_{lk}$ Zentralkraft $\Rightarrow \frac{dL}{dt} = \sum\limits_k (\vec{r}_k \times \vec{F}^e_k) = \vec{M}^e$ Drehmoment der äußeren Kraft.
$\Rightarrow \vec{L}$ erhalten, wenn $\vec{M}^e = 0$ und innere Kraft Zentralkraft.

3.3 Schwerpunkt

$\vec{R} = \frac{\sum\limits_k m_k \vec{r}_k}{\sum\limits_k m_k} = \frac{1}{M} \sum\limits_k m_k \vec{r}_k$
$\frac{d\vec{P}}{dt} = \sum\limits_i m_i \ddot{\vec{r}}_i = \frac{d}{dt}(M\dot{\vec{R}}) = \vec{F}^e$ mit $\vec{P} = M\dot{\vec{R}}$
$\vec{r} = \vec{R} - \vec{r}'$
$\vec{P} = \sum\limits_k \vec{p}_k = \sum\limits_k m_k \vec{v}_k = \sum\limits_k m_k \dot{\vec{r}}_k = \sum\limits_k m_k (\dot{\vec{R} + \vec{r}'})$
$= M\dot{\vec{R}} + \underbrace{\sum\limits_k m_k \dot{\vec{r}}'_k}_{P_{ems}} = M\dot{\vec{R}} + P_{ems}$
$\vec{L} = \sum\limits_k (\vec{r}_k \times \vec{p}_k) = \sum\limits_k ((\vec{R} + \vec{r}'_k) \times \vec{p}_k) = \vec{R} \times \vec{P} + \underbrace{\sum\limits_k \vec{r}'_k \times \vec{p}_k}_{L_{ems}}$

3.4 Kinetische Energie

$T = \sum\limits_k \frac{m_k}{2} \dot{\vec{r}}^2_k = \frac{1}{2} \sum\limits_k m_k (\dot{\vec{R}} + \dot{\vec{r}}')^2 = \frac{1}{2} \sum\limits_k m_k (\dot{\vec{R}}^2 + 2\dot{\vec{R}}\dot{\vec{r}}'_k + \dot{\vec{r}}'^2_k)$
$= \frac{1}{2} M\dot{\vec{R}}^2 + \frac{1}{2} \sum\limits_k m_k \dot{\vec{r}}'^2_k + \underbrace{\sum\limits_k m_k \dot{\vec{R}}\dot{\vec{r}}'_k}_{\dot{\vec{R}} \sum\limits_k m_k \dot{\vec{r}}'_k}$

3.5 2-Teilchen-System

SP: $\vec{R} = \frac{1}{M}(m_1\vec{r}_1 + m_2\vec{r}_2)$, $M = m_1 + m_2$, $\vec{r} = \vec{r}_2 - \vec{r}_1$
$\Rightarrow \vec{r}_1 = \vec{R} - \frac{m_2}{M}\vec{r}$, $\vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r}$
Betrachte $V(\vec{r}_1, \vec{r}_2) = V(|\vec{r}_1 - \vec{r}_2|) = V(|\vec{r}|) = V(r)$
$m_1\ddot{\vec{r}}_1 + m_2\ddot{\vec{r}}_2 = -\vec{\nabla}_1 V(r) - \vec{\nabla}_2 V(r) = 0 = m\ddot{\vec{R}}$
$\vec{\nabla}_1 V(r) = \vec{\nabla}_1 V(|\vec{r}_1 - \vec{r}_2|) = -\vec{\nabla}_2 V(r)$
$\Rightarrow$ freie Bewegung des Schwerpunktes.
$\ddot{\vec{r}} = \ddot{\vec{r}}_2 - \ddot{\vec{r}}_1 = \frac{1}{m_2}\vec{F}_2 - \frac{1}{m_1}\vec{F}_1 = -(\frac{1}{m_1} - \frac{1}{m_2})\vec{\nabla}_r V(r) = (\frac{1}{m_1} + \frac{1}{m_2})\vec{F} = \frac{m_1+m_2}{m_1 m_2}\vec{F} = \frac{1}{\mu}\vec{F}$
mit reduzierter Masse $\mu = \frac{m_1 m_2}{m_1+m_2}$
$\vec{F} = \mu\ddot{\vec{r}} = \frac{d\vec{p}}{dt} \Rightarrow \vec{p} = \mu\dot{\vec{r}} = \frac{m_1 m_2}{m_1+m_2}(\dot{\vec{r}}_2 - \dot{\vec{r}}_1) = \frac{m_1}{M}\vec{p}_2 - \frac{m_2}{M}\vec{p}_1 \neq \vec{p}_2 - \vec{p}_1$
Für $m_2 >> m_1$ gilt: $\mu = m_1$

4 Kontinuumsmechanik

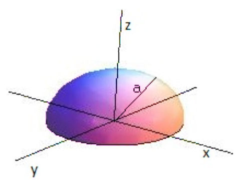
4.1 Dichten und Dichteverteilung

$\sum\limits_k m_k\vec{r}_k \approx \sum\limits_k \vec{r}_k dm_k = \underset{N\to\infty}{\longrightarrow} \int \vec{r}dm = \int \varrho(\vec{r})\vec{r}d^3r$
$M = \int dm = \int \varrho(\vec{r})d^3r$
$\int \varrho(\vec{r})\vec{r}^n d^3r$: ntes Moment der Dichteverteilung.
1. Moment: Schwerpunkt
2. Moment Abwechung vom Schwerpunkt etc.
$\int \varrho(x)dx = M$, $\int\limits_{-\infty}^{\infty} \underbrace{\varrho(x)}_{\overset{!}{=}gerade\ (\varrho(-x)=\varrho(x))} xdx = 0$
$\vec{R} = \frac{\int \varrho(\vec{r})\vec{r}d^3r}{\int \varrho(\vec{r})rd^3r}$
$\vec{P} = \sum\limits_k m_k\vec{r}_k \to \int \underbrace{\varrho(\vec{r})v(\vec{r})}_{Impulsdichte} d^3r$
$\sum\limits_k \vec{r}_k \times \vec{p}_k = \sum\limits_k m_k\vec{r}_k \times \vec{v}_k = \int \underbrace{\varrho(\vec{r})(\vec{r} \times v(\vec{r}))}_{Drehimpulsdichte} d^3r$

4.2 Beispiel zum Schwerpunkt

Halbkugel mit Radius a und konstanter Dichte ϱ_0

$\int\limits_{HK} \varrho(\vec{r})\vec{r}d^3r = \varrho_0 \int\limits_{HK} \vec{r}\vec{r}d^3r$
$\int\limits_{HK} xd^3r = \int\limits_{HK} yd^3r$
$\int\limits_{HK} zd^3r = \int\limits_0^a \int\limits_0^{2\pi} \int\limits_0^{\varrho(z)} z\varrho d\varrho d\varphi dz = 2\pi \int\limits_0^a \int\limits_0^{\varrho(z)} z\varrho d\varrho dz$, $\varrho(z) = \sqrt{a^2 - z^2}$
$... = 2\pi \int\limits_0^a \frac{\varrho^2(z)}{2} zdz = \pi \int\limits_0^a (a^2 - z^2)zdz = \pi(a^2\frac{a^2}{2} - \frac{a^4}{4}) = \frac{\pi a^4}{4}$
$M = \int\limits_{HK} \varrho(\vec{r})d^3r = \varrho_0 \int\limits_{HK} d^3r = \varrho_0 \frac{2}{3}\pi R^3$
$\Rightarrow \vec{R} = (0, 0, \frac{\frac{\pi a^4}{4}\varrho_0}{\varrho_0 \frac{2}{3}\pi R^3}) = (0, 0, \frac{3}{8}a)$

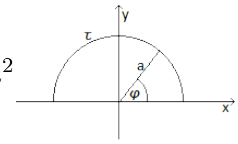


Draht in x-y-Ebene mit kleiner Dicke

Oberflächendichte: $\sigma = \frac{dm}{dF}$, Längendichte $\tau = \frac{dm}{dl}$

$MY = \int\limits_{HK} \tau y dl = \tau_0 \int\limits_{HK} y dl = \tau_0 a \int\limits_0^\pi sin(\varphi) a d\varphi = \tau_0 a^2 [-cos(\varphi)]_0^\pi = 2\tau_0 a^2$

$M = \int\limits_{HK} \tau y dl = \tau_0 \int\limits_{HK} dl = \tau_0 \pi a \Rightarrow Y = \frac{2a}{\pi}$



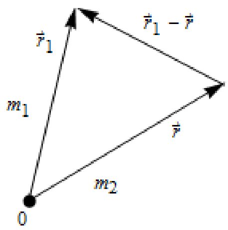
4.3 Gravitationskraft im SP

$\vec{F}_{l\to m} = G \frac{mm_1}{|\vec{r_1}-\vec{r}|^2} \frac{\vec{r_1}-\vec{r}}{|\vec{r_1}-\vec{r}|}$

$\Rightarrow F_{\to m} = Gm \sum\limits_k \frac{m_k}{|\vec{r}_k-\vec{r}|^2} \frac{\vec{r}_k-\vec{r}}{|\vec{r}_k-\vec{r}|}$

$\Rightarrow F_{\to m} = GM \int d^3r' \frac{\varrho(\vec{r}')}{|\vec{r}'-\vec{r}|^3} (\vec{r}' - \vec{r})$

$\vec{F} = -\vec{\nabla}\phi$ mit $\phi(\vec{r}) = -G \sum\limits_k \frac{mm_k}{|\vec{r}'_k - \vec{r}|} \to -Gm \int \frac{\varrho(\vec{r}')}{|\vec{r}'-\vec{r}|} d^3r'$

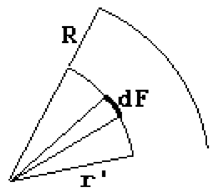


Beispiel homogene Kugel:

$\varrho(\vec{r}) = \begin{cases} \varrho 0 & r \le R \\ 0 & r > R \end{cases},$

$\int\limits_K \frac{\varrho(\vec{r}')}{|\vec{r}-\vec{r}'|} d^3r' = \varrho_0 \int\limits_K \frac{1}{|\vec{r}-\vec{r}'|} d^3r'$

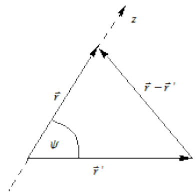
$\int \frac{1}{|\vec{r}-\vec{r}'|} r'^2 dr' \underbrace{sin(\vartheta') d\vartheta' d\varphi'}_{d\Omega} = \int \frac{1}{|\vec{r}-\vec{r}'|} dF' dr'$ (mit $d\Omega = \frac{dF}{dr^2}$)



$d\phi(\vec{r}) = -Gm\varrho_0 \int \frac{dF'}{|\vec{r}-\vec{r}'|} = -G\varrho_0 dr' \int \frac{r'^2 d\Omega}{|\vec{r}-\vec{r}'|}$

(Betrag einer Kugelschale mit Diche dr' und Radius r' zum Potential)

$\int\limits_K \frac{dF'}{|\vec{r}-\vec{r}'|} = \int\limits_K \frac{r'^2 sin(\vartheta') d\vartheta' d\varphi'}{|\vec{r}-\vec{r}'|} = \int\limits_K \frac{r'^2 sin(\vartheta') d\vartheta' d\varphi'}{(r^2+r'^2-2rr'cos(\psi))^{\frac{1}{2}}}$ ($\psi = \psi(\vartheta', \varphi')$ kompliziert!)



$\psi = \vartheta'$, wenn z-Achse$||\vec{r} \Rightarrow \int\limits_K \frac{dF'}{|\vec{r}-\vec{r}'|} = 2\pi \int\limits_0^\pi \frac{r'^2 sin(\vartheta') d\vartheta'}{(r^2+r'^2-2rr'cos(\vartheta'))^{\frac{1}{2}}}$

Substitution mit $u = cos(\vartheta') \Rightarrow du = -sin(\vartheta')$

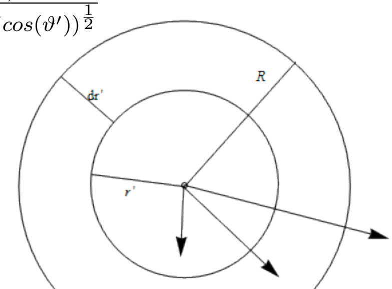
$\Rightarrow 2\pi r'^2 (-) \int\limits_1^{-1} \frac{du}{(r^2+r'^2+2rr'u)^{\frac{1}{2}}}$

Substitution mit $v = r^2 + r'^2 + 2rr'u$

$\Rightarrow 2\pi r'^2 \int\limits_{(r+r')^2}^{(r-r')^2} \frac{1}{v^{\frac{1}{2}}} dv (-\frac{1}{2rr'}) 2\pi r'^2 (-\frac{1}{2rr'}) [2v]_{(r+r')^2}^{(r-r')^2}$

$= -2\pi \frac{r'}{r} ((|r - r'| - (r + r'))$

$\Rightarrow d\phi(r) = -G\varrho_0 dr' (-2\pi) \frac{r'}{r} ((|r - r') - (r + r'))$



a) $r > r'$: $d\phi(r) = -G\frac{dM}{r}$

b) $r < r'$: $d\phi(r) = -G\frac{dM}{r'} = const$

$\phi(r) = \int d\phi$:

a) außen: $r > R$ $\quad r > r'$ für alle $r' \Rightarrow \phi(r) = -G \int \frac{dM}{r} = -\frac{GM}{r} = \frac{4}{3r} G\varrho_0 \pi R^3$

b) innen: $r < R$ $\quad \phi(r) = \int\limits_{r<r'} d\phi + \int\limits_{r>r'} d\phi = -G \int\limits_{r<r'} \frac{dM}{r'} - G \int\limits_{r>r'} \frac{dM}{r'}$

$= -G\varrho_0 \int\limits_r^R \frac{4\pi r'^2 dr'}{r'} - G\frac{1}{r} M_0$

(M_0 Masse innerhalb Kugel mit Ratius r', $M = \frac{4}{3}\pi r^3 \varrho$)

$= -4\pi G\varrho_0 (\frac{R^2}{2} - \frac{r^2}{2}) - G\frac{M_0}{r} = -4\pi G\varrho_0 \frac{1}{2}(R^2 - r^2) - \frac{4}{3}\pi r^3 G\varrho_0 \frac{1}{r}$

$= -GM\frac{1}{R}(\frac{2}{3} - \frac{1}{2}(\frac{r}{R})^2)$

$\vec{F} = -m\vec{\nabla}\phi = -m\frac{d\varphi}{dr}\vec{e}_r$

$\left(\nabla^2\phi = (\frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial z^2})\phi = \frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr} \text{ wenn } \phi = \phi(r)\right)$

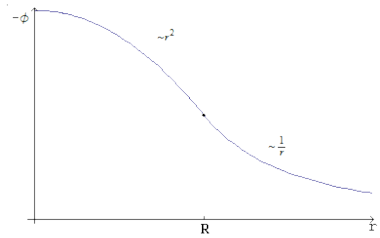
a) außen: $\nabla^2\phi = -GM\nabla^2\frac{1}{r}$

$\nabla^2\frac{1}{r} = \vec{\nabla}\vec{\nabla}\frac{1}{1} = \vec{\nabla}\vec{\nabla}\frac{1}{\sqrt{x^2+y^2+z^2}} = \vec{\nabla}(-\frac{1}{2}\frac{1}{(...)^{\frac{3}{2}}} * 2\vec{r}) = -\vec{\nabla}(\frac{1}{r^3}\vec{r})$

$\vec{\nabla}\frac{1}{r^3} = \vec{\nabla}\frac{1}{\sqrt{x^2+y^2+z^2}} = -\frac{3}{2}\frac{1}{(...)^{\frac{3}{2}}}2\vec{r} = -\frac{3}{r^5}\vec{r}$

$= -((\vec{\nabla}\frac{1}{r^3})\vec{r} + \frac{1}{r^3}(\vec{\nabla}\vec{r})) = 3\frac{r^2}{r^5} - \frac{3}{r^3} = 0$

"Laplace-Gleichung"



b) innen: $\nabla^2\phi = \vec{\nabla}\vec{\nabla}\phi = GM\frac{1}{2R^3}\vec{\nabla}\vec{\nabla}r^2$

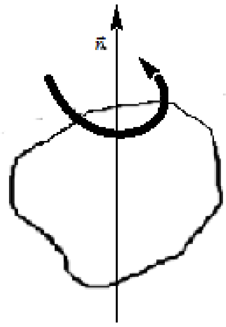
$\vec{\nabla}r^2 = \vec{\nabla}(x^2+y^2+z^2) = 2\vec{r}$

$\Rightarrow \nabla^2\phi = GM\frac{1}{2R^3}\cdot 6 = G\varrho_0\frac{4}{3}\pi R^3\frac{1}{2R^3}\cdot 6 = 4\pi\varrho_0 G$

$$\Rightarrow \nabla^2\phi = \begin{cases} 0 & r > R \\ 4\pi G\varrho_0 & r < R \end{cases}$$

$\nabla^2\phi = 4\pi G\varrho(r),\ \varrho(r) = \varrho_0\theta(r-R)$ "Poisson-Gleichung"

$\Rightarrow \phi(r) = GM\int\frac{d^3r'}{|\vec{r}-\vec{r}'|} + \phi_0(r)$ mit ϕ_0 Lösung von $\nabla^2\phi_0 = 0$



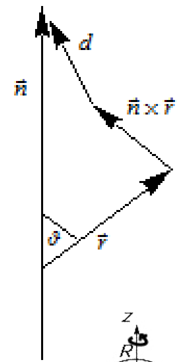
5 Bewegung starrer Körper

5.1 Drehung eines starren Körpers um feste Achse $\vec{n}$

$\vec{L} = \int\varrho(\vec{r})(\vec{r}\times\vec{v}(r))d^3r \sim n$

$L = \vec{L}\cdot\vec{n} = \int\varrho(\vec{r}\times\vec{v})d^3r\times\vec{n} = \int\varrho(\vec{n}\times\vec{r})\vec{v}d^3r = \int\varrho\underbrace{\vec{r}sin(\vartheta)}_{d}\vec{v}d^3r = \int\varrho d^2d^3r\dot{\varphi}$

$\Rightarrow \vec{L} = \underbrace{\int\varrho d^2d^3r}_{=:I}\,\vec{\omega} = I\vec{\omega}$ (I:Trägheitsmoment)

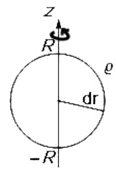


$(wd\vec{\omega} = \dot{\varphi}\vec{n})$

Kinetische Energie: $T = \frac{1}{2}\int\varrho v^2d^3r = \frac{1}{2}int\varrho(\dot{x}^2+\dot{y}^2)d^3r = \frac{1}{2}\int\underbrace{\varrho d^2d^3r}_{I} = \frac{1}{2}Iw^2$

Impuls$=mv$, $T = \frac{1}{2}mv^2 = \frac{1}{2}Iw^2$

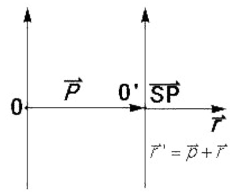
5.2 Beispiel: Trägheitsmoment einer homogenen Kugel mit ϱ_0, R



$I = \int\varrho_0 d^2d^3r = \varrho_0\int_K\varrho^2\varrho d\varrho d\varphi dz = 2\pi\varrho_0\int_{-R}^{R}dz\int_0\sqrt{R^2-z^2}\varrho^3d\varrho$

$= 2\pi\varrho_0\int_{-R}^{R}dz(R^2-z^2)^2 = \frac{\pi}{2}\varrho_0\int_{-R}^{R}dz(R^4-2R^2z^2+z^4) = \frac{\pi}{2}\varrho_0(R^42R-2R^22\frac{R^3}{3}+2\frac{2R^5}{5})$

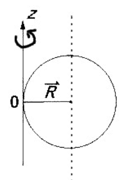
$= \frac{\pi}{2}\varrho_0(2R^5 - \frac{4}{3}R^5 + \frac{2}{5}R^5) = \frac{\pi}{2}\varrho_0\frac{16}{15}R^5 = \frac{\pi}{2}\frac{3M}{4\pi R^3}\frac{16}{15}R^5 = \frac{2}{5}MR^2$



5.3 Steinersche Satz

$\vec{r}' = \vec{p} + \vec{r} \Rightarrow I' = \int\varrho(\vec{r}')r'^2d^3r' = \int\varrho(\vec{r}')(\vec{R}+\vec{r})^2d^3r$

$= MR^2 + \underbrace{\int\varrho(\vec{r})\vec{r}^2d^3r}_{I_{SP}} + 2\vec{R}\underbrace{\int\varrho(\vec{r})\vec{r}d^3r}_{0 \text{ da SPS}} = I_{SP} + MR^2$



5.4 Beispiel: Tonne auf schiefer Ebene

Energie d. rollenden Zylinders: $E = TV = \frac{1}{2}I\omega^2 + \frac{1}{2}mv_{tr}^2 + mgz$
(I: TM der Zylinderachse)
$v_{tr} = \dot{s}$, Abrollbedingung: $v_{tr} = R\dot{\varphi} \Rightarrow E = \frac{1}{2}(I + MR^2)\dot{\varphi}^2 - MgR\varphi\sin(\alpha) + Mgh$
$\frac{dE}{dt} = 0 \Rightarrow \frac{1}{2}(I + MR^2)2\cancel{\dot{\varphi}}\ddot{\varphi} - M_0R\sin(\alpha)\cancel{\dot{\varphi}} = 0 = \ddot{\varphi} - \frac{MgR}{I+MR^2}\sin(\alpha) \Rightarrow \frac{g\sin(\alpha)}{\frac{I}{MR}+R} = \ddot{\varphi}$

5.5 Physikalisches Pendel

(Mathematisches Pendel: $\omega^2 = \frac{g}{l}$)
$M = I\ddot{\varphi} = \frac{dL}{dt} = -Mgh\sin(\varphi) \Rightarrow I\ddot{\varphi} + Mgh\sin(\varphi) = 0$
(kleine Ausschläge: $\sin(\varphi) = \varphi$, Bogenmaß!)
$\Rightarrow \ddot{\varphi} + \frac{Mgh}{I}\varphi = 0$
$\Rightarrow$ harmonischer Oszillator mit $\omega^2 = \frac{Mgh}{I}$
Länge des äquivalenten mathematischen Pendels: $l_{eq} = \frac{I}{Mh}$
Änderung von ω bei Änderung von h: $d\omega^2 = d(\frac{Mgh}{I})$, $I = I_{SP} + Mh^2$
$\Rightarrow d(\frac{Mgh}{I}) = d(\frac{Mgh}{I_{SP}+Mh^2}) = Mg\frac{(I_{SP}+Mh^2)-2Mh}{(I_{SP}+Mh^2)^2}dh$
$d\omega^2 = 0$ für $I_{SP} + Mh^2 - 2Mh^2 = 0 \Rightarrow I_{SP} = Mh^2 \Rightarrow h = \sqrt{\frac{I_{SP}}{M}}$

6 Drehung um einen Punkt

2Ks: 1. Laborsystem, 2.Körperfestes System
1. Lab: $\vec{r} = \vec{R} + \vec{r}' = \vec{R} + \sum_{i=1}^{3} x_i'\vec{e}_i'$ ($\vec{e}_i'$: karth. Einheitsvektor im körperfesten System)

$$\dot{\vec{r}} = \dot{\vec{R}} + \underbrace{\sum_i \dot{x}_i{}'\vec{e}_i}_{0 \text{ im SPS}} + \underbrace{\sum_i x_i'\dot{\vec{e}}_i'}_{\vec{\omega}\times\vec{r}'}$$

$$\dot{\vec{r}} = \dot{\vec{R}} + \dot{\vec{r}}' + \vec{\omega} \times \vec{r}'$$

6.1 Kinetische Energie

$$T = \frac{1}{2}\sum_k m_k v_k^2 = \frac{1}{2}\sum_k m_k(\dot{\vec{R}}+\vec{\omega}\times\vec{r'}_k)^2 = \frac{1}{2}M\dot{\vec{R}}^2 + \underbrace{\dot{\vec{R}}\sum_k m_k\vec{\omega}\times\vec{r'}_k}_{0 \text{ wenn in KfS, SP=0}} + \frac{1}{2}\sum_k m_k(\vec{\omega}\times\vec{r'}_k)^2$$

$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{a}\vec{c})(\vec{b}\vec{d}) - (\vec{b}\vec{c})(\vec{a}\vec{d})$
$\Rightarrow (\vec{\omega} \times \vec{r'}_k)^2 = \vec{\omega}^2\vec{r'}_k^2 - (\vec{r'}_k\vec{\omega})^2$
$T_w = \frac{1}{2}\sum_k m_k(\vec{\omega} \times \vec{r}_k')^2 = \frac{1}{3}\sum_k m_k(\vec{\omega}^2\vec{r'}_k^2 - (\vec{r'}_k\vec{\omega})^2)$
$\vec{\omega}^2\vec{r'}_k^2 - (\vec{r'}_k\vec{\omega})^2 = \sum_{i=1}^{3}\omega_i^2 {r'_k}^2 - (\sum_i x_i^{k'}\omega_i)^2 = \sum_{i=1}^{3} w_i^2 {r'_k}^2 - \sum_{ij} x_i^{k'}x_j^{k'}\omega_i\omega_j$
$= \sum_{ij}({r'_k}^2\delta_{ij} - x_i^{k'}x_j^{k'})\omega_i\omega_j$
$T_w = \frac{1}{2}\sum_k m_k\sum_{ij}({r'_k}^2\delta_{ij} - x_i^{k'}x_j^{k'})\omega_i\omega_j = \frac{1}{2}I_{ij}\omega_i\omega_j$
$I_{ij} = \sum_k m_k\sum_{ij}({r'_k}^2\delta_{ij} - x_i^{k'}x_j^{k'})$
Kontinuumsdarstellung: $I_{ij} = \int \varrho(\vec{r})(\vec{r}^2\delta_{ij} - x_i^{k'}x_j^{k'})d^3r$

(3 × 3-Matrix, Trägheitstensor)

$T = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\sum_{ij} I_{ij}\omega_i\omega_j$

$$I = \begin{pmatrix} \int \varrho(y^2+z^2)d^3r & -\int \varrho xyd^3r & -\int \varrho xzd^3r \\ -\int \varrho yxd^3r & \int \varrho(x^2+z^2)d^3r & -\int \varrho yzd^3r \\ -\int \varrho zxd^3r & -\int \varrho zyd^3r & \int \varrho(x^2+y^2)d^3r \end{pmatrix}$$

(Trägheitstensor, Symmetrisch, 6 unabhängige Komponenten)

6.2 Drehimpuls des starren Körpers

$\vec{L} = \int \varrho(\vec{r}\times\vec{v})d^3r = \int \varrho(\vec{r}\times(\vec{\omega}\times\vec{r}))d^3r = \int \varrho[\vec{\omega}r^2 - \vec{r}(\vec{\omega}\cdot vvr)]d^3r$

$L_i = \int \varrho(r^2\omega_i - x_i\sum_j \omega_j x_j)d^3r = \sum_j \int \varrho(r^2\delta_{ij} - x_i x_j)d^3r = I_{ij}\omega_j \Rightarrow \vec{L} = I\vec{\omega}$

$\vec{L}$ im allgemeinen nicht nicht parallel zu $\vec{\omega}$

6.3 Bestimmung der Hauptträgheitsachsen

Angenommen es gibt körperfestes System, sodass $\vec{L}||\vec{\omega} \Rightarrow \vec{L} = I_o\vec{\omega}$ mit I_0 Zahl.

$L_i = I_{ij}\omega_j = I_0\omega_i \Rightarrow I_{ij}\omega_j - I_0\omega_i = (I_{ij} - I_0\delta_{ij})\omega_j = 0$

(Summe über j: 1 bis 3 [Einstein-Summen-Konvention])

Lineares homogenes 3d. Gleichungssystem für Unbekannte: $\omega_1, \omega_2, \omega_3$

Nichttriviale Lösung$\Leftrightarrow$ $|I_{ij} - I_0\delta_{ij}| = 0 \Rightarrow I_0$ Eigenwert des Trägheitstensor.

3 Eigenwerte $I_0^k \Rightarrow (I_{ij} - I_0^k d_{ij})\omega_j^k = 0$ nicht trivial lösbar für $\vec{\omega}^k$ (Eigenvektor von I)

$\Rightarrow$ Trägheitstensor diagbar durch Achsenwahl $||\vec{\omega}^k$

$$\Rightarrow I = \begin{pmatrix} \int \varrho(y^2+z^2)d^3r & 0 & 0 \\ 0 & \int \varrho(x^2+z^2)d^3r & 0 \\ 0 & 0 & \int \varrho(x^2+y^2)d^3r \end{pmatrix}$$

Aber: Sind ω^i senkrecht aufeinander? Sonst kein karth. Koordsys.

$$\left.\begin{array}{ll} I\vec{\omega}^k = I_0\vec{\omega}^k & |\cdot\vec{\omega}^l \\ I\vec{\omega}^l = I_0\vec{\omega}^l & |\cdot\vec{\omega}^k \end{array}\right\} - \Rightarrow \vec{\omega}^l I\vec{\omega}^k - \vec{\omega}^k I\vec{\omega}^l = (I_0^k - I_0^l)\vec{\omega}^k\vec{\omega}^l$$

$$= \vec{\omega}_i^l I_{ij}\vec{\omega}_j^k - \vec{\omega}_j^k I_{ji}\vec{\omega}_i^l = \underbrace{(I_{ij} - I_{ji})}_{=0,\ da\ diagonal} \Rightarrow (I_0^k I_0^l)\vec{\omega}_k\vec{\omega}_l = 0$$

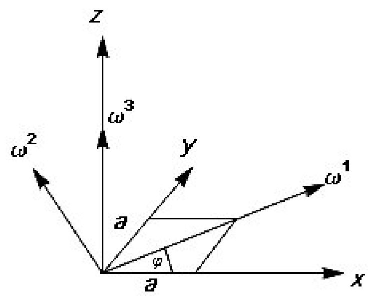
$\Rightarrow \vec{\omega}_i$ orthogonal aufeinander, wenn alle Eigenwerte verschieden.

6.4 Beispiel: quadratische Platte in x,y-Ebene

Seitenlänge a, keine z-Ausdehnung, daher σ_0 :Flächendichte.

$$I_{ij} = \int \sigma_0(r^2\delta_{ij} - x_i x_j)dF \Rightarrow \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_0\int y^2dF & -\sigma_0\int xydF & -\sigma_0\int xzdF \\ I_{21} = I_{12} & \sigma_0\int x^2dF & -\sigma_0\int yzdF \\ I_{31} = I_{13} & I_{32} = I_{23} & \sigma_0\int(x^2+y^2)dF \end{pmatrix}$$



$$= \begin{pmatrix} \sigma_0\frac{a^4}{3} & -\sigma_0\frac{a^4}{4} & 0 \\ -\sigma_0\frac{a^4}{4} & \sigma_0\frac{a^4}{3} & 0 \\ 0 & 0 & \frac{2}{3}\sigma_0 a^4 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}Ma^2 & -\frac{1}{4}Ma^2 & 0 \\ -\frac{1}{4}Ma^2 & \frac{1}{3}Ma^2 & 0 \\ 0 & 0 & \frac{2}{3}Ma^2 \end{pmatrix}$$

Nun sei $|I| \stackrel{!}{=} 0$, also

$(\frac{1}{3}Ma^2 - I_0)^2(\frac{2}{3}Ma^2 - I_0) - (\frac{1}{4}Ma^2)^2(\frac{2}{3}Ma^2 - I_0) = 0$

$I_{0,3} = \frac{3}{4}Ma^2 \Rightarrow (\frac{1}{3}Ma^2 - I_0)^2 - (\frac{1}{4}Ma^2)^2 = 0 \Rightarrow I_{0,1} = \frac{1}{12}Ma^2,\ I_{0,2} = \frac{7}{12}Ma^2$

Gleichungssystem für ersten Eigenwert: $(I_{ij} - I_0\delta_{ij})\omega_j = 0$

$(\frac{1}{3}Ma^2 - \frac{1}{12}Ma^2)\omega_1^1 - \frac{1}{4}Ma^2\omega_2^1 = 0 \Rightarrow \omega_1^1 = \omega_2^1$

$-\frac{1}{4}Ma^2\omega_1^1 + (\frac{1}{3}Ma^2 - kr12Ma^2)\omega_2^1 = 0 \Rightarrow \omega_1^1 = \omega_2^1$

$(\frac{2}{3}Ma^2 - \frac{1}{12}Ma^2)\omega_3^1 = 0 \Rightarrow \omega_3^1 = 0$

$\Rightarrow \vec{\omega}^1 = (\omega_0, \omega_0, 0)$

2. Eigenwert:

$\omega_1^2 = -\omega_2^2, \omega_3^2 = 0 \Rightarrow vv\omega^2 = (\omega_0, -\omega_0, 0)$

3. Eigenwert:

$-\frac{1}{3}\omega_1^3 - \frac{1}{4}\omega_2^3 = 0,\ -\frac{1}{4}\omega_1^3 - \frac{1}{3}\omega_2^3 = 0 \Rightarrow \omega_1^3 = \omega_2^3 = 0$

$(\frac{2}{3} - \frac{2}{3})\omega_3^3 = 0 \Rightarrow \vec{\omega}^3 = (0, 0, \omega_0)$

Also Achsen orthogonal zueinander!

Drehung um eine w_i-Achse: $\vec{\omega}||\vec{L}$ I im neuen Koordinatensystem diagonalisiert!

6.5 Rotation eines starren Körpers um einen Festen Punkt

$T = \frac{1}{2}\sum\limits_j I_{ij}\omega_i\omega_j;\quad \vec{I} = I\vec{\omega};\quad (L_i = I_{ij}\omega_j)\quad I_{ij} = \int \varrho(\vec{r})(r^2\delta_{ij} - x_i x_j)d^3r$ (KF Koord.)

$\vec{L}||\vec{\omega} \Rightarrow I_{ij} = I_0\omega_i = L_i$ nur nichttriviale Lösung, wenn I_0 EW von I ist!

Drehung um Achsen, die parallel sind zu den EV von I: $\vec{L}||\vec{\omega} \Rightarrow$ „Hauptträgheitsachsen“

I_{ij} reell, symmetr. $\Rightarrow$ diagbar. durch geeign. Wahl des Koordinatensystems.

$I' = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$ (I_i: Eigenwerte von I_{ij}, Hauptträgheitsmomente).

Neues Koordinatensystem muss Achsen parallel zu $\vec{\omega}^i$ haben!

6.6 Beispiel: Fortsetzung Platte

Eigenwerte einer Matrix ändern sich nicht durch Transformation!

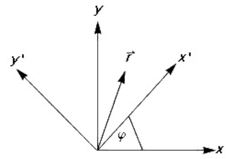
(im Körperfesten System)

Nach Drehung ist $I = (...) \Rightarrow I' = Ma^2 \begin{pmatrix} \frac{1}{12} & 0 & 0 \\ 0 & \frac{7}{12} & 0 \\ 0 & 0 & \frac{2}{3} \end{pmatrix}$

(Diagonalelemente/Eigenwerte I_i heißen Hauptträgheitsmomente)

Symmetrieachsen sind immer auch Hauptträgheitsachsen!

6.7 Koordinatentransformation



Rotation des KS: $\vec{r}(x,y) \to \vec{r'}(x',y')$, $\vec{r'} = A\vec{r}, x'_i = A_{ij}x_j$

Immer gilt: $r^2 = r'^2$, $r'^2 = \sum\limits_{i=1}^{3} x_i'^2 = \sum\limits_i (\sum\limits_j A_{ij}x_i x_j) = \sum\limits_i \sum\limits_j \sum\limits_k A_{ij}x_j A_{ik}x_k$

$= \sum\limits_{ijk} A^T_{ij}A_{ik}x_j x_k = \sum\limits_{jk}(A^TA)_{jk}x_jx_k \stackrel{1}{=} \sum\limits_k x_k^2$, also $(A^TA)_{jk} \stackrel{!}{=} \delta_{jk} \Rightarrow A^TA = \mathbb{1}$

$\Rightarrow A^T = A^{-1} \Rightarrow$ A orthogonal. Drehung ist orthogonal: $A^TA = AA^T$

$I'_{ij} = \int \varrho(r^2\delta_{ij} - x'_i x'_j)d^3r' = \int \varrho(r^2\delta_{ij} - A_{ik}x_kA_{jl}x_l)d^3r = \int \varrho(r^2(A^TA) - A_{ik}x_kA_{jl}x_l)d^3r =$
$\int \varrho(r^2A_{il}A_{lj} - A_{ik}A_{jl}x_kx_l)d^3r = \int \varrho(r^2A_{il} - A_{ik}x_kx_l)d^3r = \int \varrho(r^2\sum\limits_k A_{ik}\delta_{kl} - \sum\limits_k A_{ik}x_kx_l)d^3rA^T_{lj} =$
$A_{ik}\int \varrho(r^2\delta_{kl} - x_kx_l)d^3rA^T_{lj} = A_{ik}I_{kl}A^T_{lj} = (AIA^T)_{ij}$

Forderung: I' diagonal $\Rightarrow (AIA^T)_{ij} \stackrel{!}{=} I_i\delta_{ij} \Rightarrow$ Bestimmungsgleichung für A.

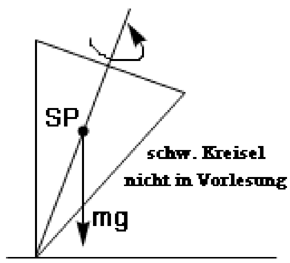
6.8 Beispiel: Fortsetung Platte

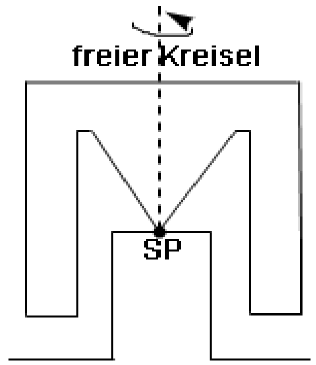
$$A = \begin{pmatrix} \cos\varphi & \sin(\varphi) & 0 \\ -\sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ für Drehung um Winkel } \varphi$$

$x' = A1jx_j = c\cos(\varphi) + y\sin(\varphi)$, $y' = A_{2j}x_j = -x\sin(pp) + y\cos(pp))$, $z' = z$

$$AIA^T = \begin{pmatrix} \cos(\varphi) & \sin(\varphi) & 0 \\ -\sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{1}{4} & 0 \\ -\frac{1}{4} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

$\Rightarrow \varphi = 45°$





7 Kreisel

7.1 Bewegungsgleichung

$\vec{M} = \frac{d\vec{L}}{dt} = \frac{d}{dt}(I\vec{\omega})$ im Labor.

$\vec{L} = \sum\limits_i L_i\vec{e}_i$, $\frac{d\vec{L}}{dt} = \sum\limits_i \frac{dL_i}{dt}\vec{e}_i$ (Labor)

$\vec{L} = \sum\limits_i L_i\vec{e}'_i$, $\frac{d\vec{L}}{dt} = \underbrace{\sum\limits_i \frac{dL_i}{dt}\vec{e}'_i}_{\frac{d'\vec{L}}{dt}} + \underbrace{\sum\limits_i L_i\frac{d\vec{e}'_i}{dt}}_{\vec{\omega}\times\vec{L}} = \frac{d'\vec{L}}{dt} + \vec{\omega}\times\vec{L}$ (Körperfestes System)

$\vec{M} = \frac{d\vec{L}}{dt} = \underbrace{\frac{d}{dt}(I\vec{\omega})}_{Labor} = \underbrace{\frac{d'\vec{L}}{dt} + \vec{\omega} + vvL}_{KS}$

Körperfestes System (KS): $M_i = \frac{d'L_i}{dt} + (\vec{\omega} + \vec{L})_i$

Wähle KS so, dass Hauptachsen. $\Rightarrow$ I diagonal $\Rightarrow L_i = I_{ij}\omega_j$ (I_{ij} zeitl. konstant im KS)

$\Rightarrow M_i = I'_{ij}\frac{d\omega_j}{dt} + \varepsilon_{ijk}\omega_jL_k$

$I_{11} = A,\ I_{22} = B,\ I_{33} = C,\ \vec{\omega} = (p,q,r)$ (Euler)

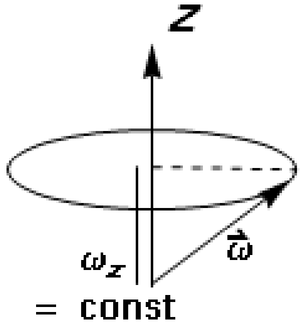
$M_1 = A\dot{p} + qL_3 - rL_2 = A\dot{p} + (C-B)qr,$

$M_2 = B\dot{q} + (A-C)rp,$

$M_3 = C\dot{r} + (B-a)pq$

(Eulerische Kreiselgleichung)

7.2 Kräftefreier, symmetrischer Kreisel



Kräftefrei: $\vec{M} = 0$, symmetrisch: $A = B$

$1) A\dot{p} + (C - A)qr = 0, \qquad 2) A\dot{q} + (A - C)rp = 0, \qquad 3) C\dot{r} + 0 = 0$

$(3) \Rightarrow \frac{d}{dt}(L_z) = 0 \Rightarrow L_z = const,\ \omega_z = const.$

x,y,z: KS, X,Y,Z:Labor

$(1\&2) \Rightarrow \dot{p} = \frac{A-C}{A}qr, \qquad (2) \Rightarrow \dot{q} = -\frac{A-C}{A}rp$

$(1\&2) \Rightarrow p\dot{p} + q\dot{q} = 0 \Rightarrow \frac{d}{dt}(p^2 + q^2) = 0 \Rightarrow \omega_y^2 + \omega_x^2$ zeitl. konstant

$\Rightarrow \omega_x^2 + \omega_y^2 + \omega_z^2 = \vec{\omega}^2 = const\ (1) \Rightarrow \ddot{p} = \frac{A-C}{A}\dot{q}r = -\frac{A-C}{A} \cdot \frac{A-C}{A}r^2 p$

$(2)\ddot{p} + (\frac{A-C}{A}r)^2 p = 0$

$p(t) = p_0 \sin(\Omega t), \quad \Omega = \frac{A-C}{A}r, \qquad q(t) = p_0 \cos(\Omega t) \Rightarrow$ glm. Präzession von $\vec{\omega}$ um die Figurenachse z mit Winkelgeschwindigkeit Ω.

Problem: Bewegung im Labor?

$r = const, \quad p^2 + q^2 = const \Rightarrow A(p^2 + q^2) + Cr^2 = const \Rightarrow I_{XY}(\omega_X^2 + \omega_Y^2) = const$

$\Rightarrow T = \frac{1}{2}I_{ij}\omega_i\omega_j, \quad \omega_j = \frac{1}{2}\vec{L}\vec{\omega}$, dann $L_i = I_{ij}\omega_j \Rightarrow \vec{L}\vec{\omega} = const$

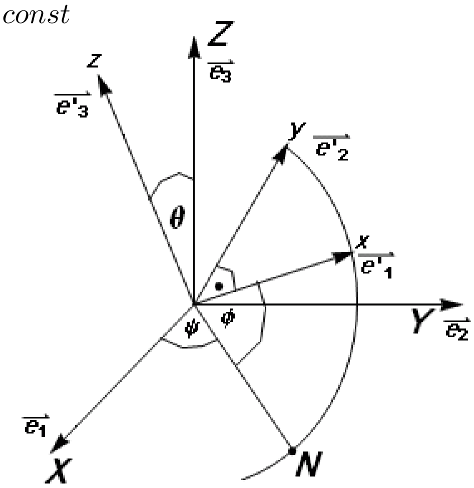
$\Rightarrow$ freier Kreisel: $\vec{M} = 0 \Rightarrow \vec{L}$ konstant im Labor, (Labor so, dass $\vec{L}||\vec{e_z}$)

$\Rightarrow \vec{L}\vec{\omega} = const, \quad \vec{L} = const \Rightarrow |\vec{L}| = const, \quad p^2 + q^2 + r^2 = \omega^2 = const$

$\Rightarrow |\omega| = const \Rightarrow \vec{L}\vec{\omega} = const$

Beispiel: $\frac{A-C}{A} = \frac{1}{300} \Rightarrow T = \frac{2\pi}{\Omega} = 2\pi \cdot \frac{300}{r} = 2\pi \cdot 300 \cdot \frac{d}{2\pi} = 300d$

(Erde dreht um Figurenachse)



7.3 Eulerische Winkel

1) Drehung um z-Achse um ψ

2) Drehung von 0-N-Achse (Knotenlinie) um θ

3) Drehung im neuen z-Achse um ϕ

$R(\psi, \theta, \phi) = R_Z(\phi)R_N(\theta)R_Z(\psi)$

$\vec{r}' = R(\psi, \theta, \phi)\vec{r}$. Momentane Drehachse: $\vec{\omega} = \vec{\omega}_1 + \vec{\omega}_2 + \vec{\omega}_3$

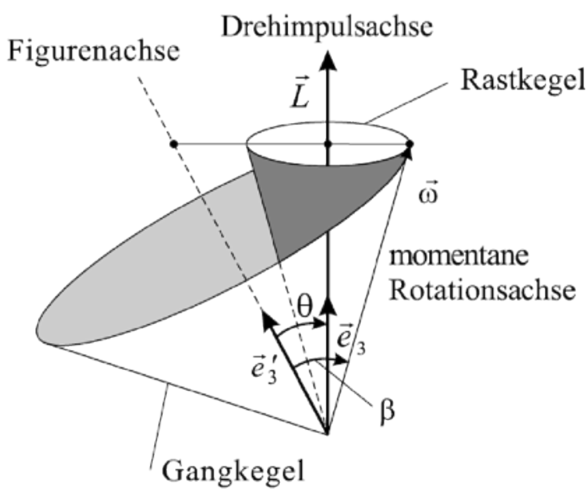
($\vec{\omega}_i$ = Drehung um Euler-Achse i)

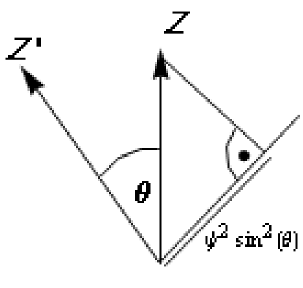
1) $\dot{\psi} : p = \dot{\psi}\sin(\theta)\sin(\phi), \quad q = \dot{\psi}\sin(\theta)\cos(\phi), \quad r = \dot{\psi}\cos(\theta)$

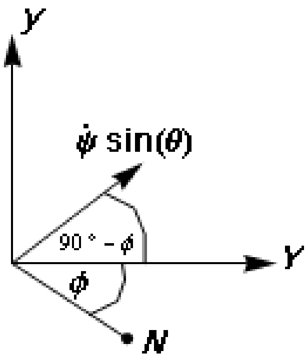
2) $\dot{\theta} : p = \dot{\theta}\cos(\phi), \quad q = -\dot{\theta}\sin(\phi), \quad r = 0$

3) $\dot{\phi} : p = 0, \quad q = 0, \quad r = \dot{\phi}$

$\Rightarrow$ Gesamtvektor $\vec{\omega} : p = \dot{\theta}\cos(\phi) + \dot{\psi}\sin(\theta)\sin(\theta), \quad q = -\dot{\theta}\sin(\phi) + \dot{\psi}\sin(\theta)\cos(\phi), \quad r = \dot{\psi}\cos(\theta)$



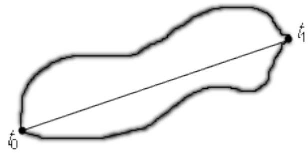




7.4 Anwendung auf Kreisel

$\vec{e}_z||\vec{L} \Rightarrow \vec{L}_z = lr = const \Rightarrow \cos(\theta) = const$
$p = \dot{\psi}\sin(\theta)\cos(phi) = \omega_\perp \sin(\Omega t),\ \ q = \dot{\psi}\sin(\theta)\cos(\phi) = \omega_\perp \cos(\Omega t),\ \ r = \psi\cos(\theta) + \dot{\phi}$
(Nur für diese Art von Kreiseln!)
$\Omega = \frac{A-C}{A},\ \ p^2 + q^2 = \omega_\perp^2 = \dot{\psi}^2 \sin^2(\theta) = const \Rightarrow \dot{\psi} = \frac{\omega_b ot}{\sin(\theta)} = const$
$\Rightarrow \dot{\psi} = \omega t - \frac{\sqrt{1+\tan^2(\theta)}}{\tan(t)} = \frac{1}{A}\sqrt{(lr)^2 + (A\omega_z)^2},\ \ \frac{p}{4=\tan(\phi)=\tan(\Omega}t) \Rightarrow \phi = \Omega t,\ \ \dot{\phi} = \Omega$

8 Lagrange-Theorie



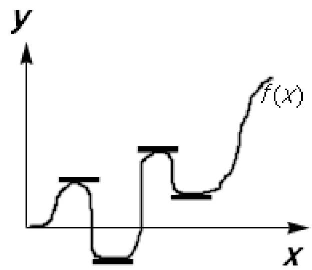
$\vec{F} = m\ddot{\vec{x}} \Rightarrow Fx = m\ddot{x}, \quad F_r = m(\ddot{r} - r\dot{\vartheta}^2 - r\sin^2(\vartheta)\dot{\varphi})$

1) Gleiche Gestalt für alle Koordinanten
2) Generalisierte Koordinaten q_i, so dass zwischen diesen keine Zwangsbedingung existiert.

(Zwangsbedingung: e.g: $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$)
$\Rightarrow$ Lagrange Fu' $L(q_i, \dot{q}_i)$, $S = \int\limits_{t_0}^{t_1} L(q_i(t), \dot{q}_i(t))dt$ (Wirkung)

8.1 Hamilt'sches Prinzip

S ist stationär für die wahre Bahn.
Bild: $\frac{df}{dx} = 0$ an stationären Punkten. $f(x) = f(x_i) + (x - x_i)\frac{df}{dx}|_{x_i} + \frac{1}{2}(x - x_i)^2 \frac{d^2 f}{dx^2}|_i$
$\Rightarrow f$ ändert sich kaum, daher stationär.
Hat man die wahre Bahn bereits gefunden, ändern kleine Änderungen an dieser Bahn S nicht!



$\delta S = \int\limits_{t_0}^{t_1} L(q_i + \delta q_i, \dot{q}_i + \delta\dot{q}_i)dt - \int L(q_i - \dot{q}_i)dt \stackrel{!}{=} 0$ für wahre Bahn.
$L(q_i + \delta q_i, \dot{q}_i + \delta\dot{q}_i) = L(q_i, \dot{q}_i) + \frac{dL}{dq_i}\delta q_i + \frac{dL}{d\dot{q}_i}\delta\dot{q}_i + ...$
$\Rightarrow \delta S = \int\limits_{t_0}^{t_1} (\frac{dL}{dq_i}\delta q_i + \frac{dL}{d\dot{q}_i}\delta\dot{q}_i)dt, \quad \delta\dot{q}_i = \delta\frac{dq_i}{dt} = \frac{d}{dt}(\delta q_i)$
$\Rightarrow \delta S = \int\limits_{t_0}^{t_1} (\frac{dL}{dq_i}\delta q_i + \frac{dL}{d\dot{q}_i}\frac{d}{dt}\delta q_i)dt = \int\limits_{t_0}^{t_1} (\frac{Dl}{dq_i}\delta q_i + \frac{d}{dt}(\frac{dL}{d\dot{q}_i}\delta q_i) - \frac{d}{dt}(\frac{dL}{dq_i}) \cdot \delta q_i)dt$
$= \int\limits_{t_0}^{t_1} (\frac{dL}{dq_i} - \frac{d}{dt}(\frac{dL}{d\dot{q}_i}))\delta q_i dt + \int\limits_{t_0}^{t_1} \frac{d}{dt}(\frac{dL}{d\dot{q}_i}dq_i)$
Fixierter Anfangs und Endpunkt: $\delta q_i(t_0) = \delta q_i(t_1) = 0$
$\Rightarrow \delta S = \int\limits_{t_0}^{t_1} (\frac{dL}{dq_i} - \frac{d}{dt}(\frac{dL}{d\dot{q}_i})\delta q_i)dt \stackrel{!}{=} 0 \Rightarrow \underbrace{\frac{d}{dt}(\frac{dL}{d\dot{q}_i}) - \frac{dL}{dq_i}}_{\text{Lagrangesche Bewegungsgl.}} = 0$, da δq beliebig.
$F = \frac{dp}{dt} \Rightarrow \frac{dp}{dt} - F = 0$
konservative Kraft: $F = -\frac{dV}{dx} \Rightarrow \frac{dp}{dt} - (-\frac{dV}{dx}) = 0 = \frac{d}{dt}m\dot{x} - (-\frac{dV}{dx}) = 0 = \frac{d}{dt}\frac{d}{dx}(\frac{1}{2}m\dot{x}^2) - (-\frac{dV}{dx}) = 0 \Rightarrow L = T - V = \frac{1}{2}m\dot{x}^2 - V$
allgemeine Lagrange-Funktion
$\Rightarrow L(q_i, \dot{q}_i) = T - V$
Zwangsbedingungen: $F_k(x_1, ..., x_n, t) = 0$ heißen holonone ZB.
Beispiel: $dZ. \sum\limits_i (x_i^1 - x_i^2)^2 = 0$

Nicht holonone ZB sind beispielsweise Ungleichungen.

$F_k(x_1, ..., t) = 0$ heißen auch rehonone, $F_k(x_1, ..., x_n) = 0$ skleronone ZB.

8.2 Holonone Zwangsbedingungen

$q_i = q_i(x_1, ..., x_n, t)$, $i = 1, ..., f$, $N - f$: Zahl der ZB., $x_i = x_i(q_1, ..., q_n, t)$

$$T = \frac{1}{2}\sum\limits_k m_k \dot{x}_k^2 = \frac{1}{2}\sum\limits_k m_k(\frac{dx_k}{dq_l}\dot{q}_l\frac{dx_k}{dt})^2 = \frac{1}{2}\sum\limits_k m_k(\sum\limits_{lj}\frac{dx_k}{dq_l}\frac{dx_k}{dq_j}\dot{q}_l\dot{q}_j + 2\sum\limits_l \frac{dx_l}{dq_l}\dot{q}_l\frac{dx_k}{dt} + (\frac{dx_k}{dt})^2)$$

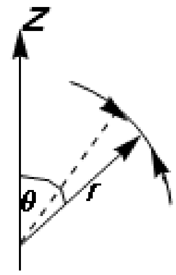
8.3 Skleronone Zwangsbedingungen

$$T = \frac{1}{2}\sum\limits_k mk\sum\limits_{lj}\frac{dx_k}{dq_l}(q_i)\frac{dx_k}{dq_j}(q_i)\dot{q}_l\dot{q}_j = \sum\limits_{lj}\sum\limits_k \frac{m_k}{2}a_{kl}(q_i)a_{kj}(q_i)\dot{q}_l\dot{q}_j$$

$$T = \frac{1}{2}\sum\limits_{lj} m_{lj}(q_i)\dot{q}_l\dot{q}_j \Rightarrow \frac{d}{dt}(\frac{dL}{d\dot{q}_i}) - \frac{dL}{dq_i} = \frac{d}{dt}\sum\limits_j(m_{ij}\dot{q}_j) - \frac{dT}{dq_i} + \frac{dV}{dq_i}$$

$$= \sum\limits_j m_{ij}\ddot{q}_j - \frac{1}{2}\sum\limits_{lj}\frac{dm_{lj}}{dq_i}\dot{q}_l\dot{q}_j + \frac{dV}{dq_i} = 0, \quad i = 1, ..., f, \quad V = V(q_1, ..., q_f)$$

8.4 Betrachte: kart. Koordsys. zu Kugelkoordsys.



kart: $m\ddot{x}_i = F_i - \frac{dV}{dx_i}$

Kugel:

$$m(\ddot{r} - r\dot{\vartheta}^2 - r\sin^2(\vartheta)\dot{\varphi}^2) + \frac{\partial V}{\partial r} = 0$$
$$mr(r\ddot{\vartheta} + 2\dot{r}\dot{\vartheta} - r\sin(\vartheta)\cos(\vartheta)\dot{\varphi}^2) + \frac{\partial V}{\partial\vartheta} = 0$$
$$mr\sin(\vartheta)(r\sin(\vartheta)\ddot{\varphi} + 2\sin(\vartheta)\dot{r}\dot{\varphi} + 2r\cos(\vartheta)\dot{\vartheta}\dot{\varphi}) + \frac{\partial V}{\partial\varphi} = 0$$

$T = \frac{1}{2}mv^2 = \frac{m}{2}(\frac{ds}{dt})^2$, $\quad ds^2 = dr^2 + r^2 d\vartheta^2 + r^2\sin^2(\vartheta)\partial\varphi^2$

$v^2 = (\frac{ds}{dt})^2 = \dot{r}^2 + r^2\dot{\vartheta}^2 + r^2\sin(\vartheta)\dot{\varphi}^2$

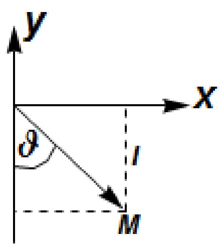
$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\vartheta}^2 + r^2\sin^2(\vartheta)\dot{\varphi}^2) - V(r, \vartheta, \varphi)$

$\frac{\partial L}{\partial\dot{q}} = \frac{\partial L}{\partial\dot{r}} = m\dot{r} \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial\dot{q}} - \frac{\partial L}{\partial q} = \frac{d}{dt}(m\dot{r}) - \frac{\partial L}{\partial r}) = m\ddot{r} - mr\dot{\vartheta}^2 - mr\sin^2(\vartheta)\dot{\varphi} + \frac{\partial V}{\partial r} = 0$

$L = T - V = L(q_i, \dot{q}_i, t)$; $\quad T = \frac{1}{2}\sum\limits_{ij} m_{ij}(q_k)\dot{q}_i\dot{q}_j$

$\delta\int\limits_{t_0}^{t_1} Ldt = 0 \Rightarrow \frac{d}{dt}(\frac{prL}{pr\dot{q}_i}) - \frac{prL}{dq} = 0$ (Lagrange Bewegungsgleichung)

8.5 Beispiel: mathematisches Pendel



gesucht: generalisierte Koordinaten.

$x = l\sin(\varphi)$, $\; y = -l\cos(\varphi)$

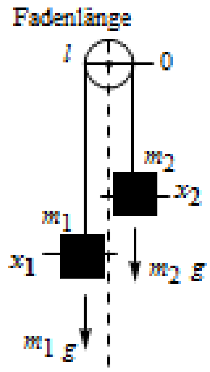
$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) = \frac{m}{2}l^2\dot{\varphi}^2 = \frac{1}{2}I\omega^2$, $\quad V = mg(y + l) = -mgl\cos(\varphi) + mgl$

$L = T - V = \frac{m}{2}l^2\dot{\varphi}^2 + mgl\cos(\varphi) - mgl \Rightarrow \varphi = q$: generalisierte Koordinate.

$\frac{\partial L}{\partial\dot{\varphi}} = ml^2\dot{\varphi}$; $\quad \frac{\partial L}{\partial}\varphi = -mgl\sin(\varphi) \Rightarrow ml^2\ddot{\varphi} + mgl\sin(\varphi) = 0 \Rightarrow \ddot{\varphi} + \frac{g}{l}\sin(\varphi) = 0$

φ klein: $\sin(\varphi) \approx \varphi \Rightarrow \varphi = \varphi_0\sin(\omega t + \delta)$, $\quad \omega^2 = \frac{g}{l}$

8.6 Atwood'sche Maschine (Umlenkrollen-Schwingung)



Nebenbedingung: konstante Fadenlänge: $x_1 + x_2 = -l$ (holonom, skleronom)
$L = T - V = \frac{1}{2}m_1\dot{x}_1^2 - \frac{1}{2}m_2\dot{x}_2^2 - m_1 g x_1 - m_2 g x_2$
$= \frac{m_1}{2}\dot{x}_1^2 + \frac{m_2}{2}(\dot{-l - x_1})^2 - m_1 g x_1 - m_2 g(-l - x_1) \Rightarrow x_1 = q$: gen. Koord.
$\frac{\partial L}{\dot{x}_1} = m_1\dot{x}_1 + m_2\dot{x}_1 \Rightarrow (m_1 + m_2)\ddot{x}_1 - m_1 g - m_2 g = 0 \Rightarrow \ddot{x}_1 = \frac{m_2 - m_1}{m_1 + m_2} g$
Alternative über Newton (Seilspannung):
$m_1\ddot{x}_1 = -m_1 g + s, \quad m_2\ddot{x}_2 = -m_2 g + s, \quad \ddot{x}_1 = -\ddot{x}_2 \Rightarrow ...$

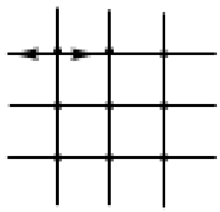
8.7 Symmetrien und Erhaltungssätze

1) L nicht explizit q-abh. $\Rightarrow \frac{\partial L}{\partial q} = 0 = \frac{d}{dt}(\frac{\partial L}{d\dot{q}}) = 0 \Rightarrow p = \frac{\partial L}{\partial \dot{p}} = const =:$ generalisierter Impuls

2) L nicht explizit von t abgh.$\Rightarrow \frac{dL}{dt} = \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} + \frac{dL}{dt} = \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}})\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} = \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}}\dot{q}) = \frac{d}{dt}(p\dot{q}) \Rightarrow \frac{d}{dt}(p\dot{q} - L) = 0 \Rightarrow p\dot{q} - L$ (Erhaltungsgröße Energie!)
$m\dot{x}\dot{x} - \frac{m}{2}\dot{x}^2 + V = \frac{m}{2}\dot{x}^2 + V = E$

Allgemein:
$p_i = \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} = \frac{\partial}{\partial \dot{q}_i}\sum_{kl} m_{kl}\dot{q}_i\dot{q}_l = \sum_{kl}\frac{m_{kl}}{2}\frac{\partial}{\partial \dot{q}_i}(\dot{q}_k\dot{q}_l) = \sum_{kl}\frac{m_{kl}}{2}(\delta_{ik}\dot{q}_l + \dot{q}_k\delta_{il})$
$= \sum_l \frac{m_{il}}{2}\dot{q}_l + \sum_k \frac{m_{kj}}{2}\dot{q}_k = \sum_l m_{il}\dot{q}_l$
$\sum_i p_i\dot{q}_i - L = 2T - (T - V) = T + V = E$

9 Normalschwingungen



Sei $x_i = \tilde{q}_i - \tilde{q}_{i0}$ gen. Koord.
$V(\tilde{q}_i) = V(\tilde{q}_{i0}) + \sum_i x_i \frac{\partial V}{\partial \tilde{q}_i}\Big|_{\tilde{q}_{i0}} + \frac{1}{2}\sum_{ij} x_i x_j \underbrace{\frac{\partial^2 V}{\partial \tilde{q}_i \tilde{q}_j}\Big|_{\tilde{q}_{i0}}}_{V_{ij}} + ...$
Im Gleichgewicht ist V minimal $\Rightarrow \frac{\partial V}{\partial \tilde{q}_i}\Big|_{\tilde{q}_{i0}} = 0 \Rightarrow V(\tilde{q}_i) = V_0 + \frac{1}{2}\sum_{ij} V_{ij} x_i x_j = V(x_i)$
$T = \frac{1}{2}\sum_{ij} m_{ij}(x_k) x_i x_j \Rightarrow L = T - V = \frac{1}{2}\sum_{ij} m_{ij}\dot{x}_i\dot{x}_j - \frac{1}{2}\sum_{ij} v_{ij} x_i x_j, \quad i, j = 1, ..., f$
Bewegungsgleichung für k-te Koordinaten:
$\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}_k}) + \frac{\partial V}{\partial x_k} = \delta = \frac{d}{dt}(\sum_j m_{jk} x_j) + \frac{\partial V}{\partial x_k} = \sum_j m_{jk}\ddot{x}_j + \frac{\partial V}{\partial x_k} = \sum_j m_{jk} x_j + \sum_j v_j x_j$
$= \sum_j (m_{jk}\ddot{x}_j + V_{jk} x_j) = 0, \quad k = 1, ..., f$
Ansatz: $x_j = A_j e^{i\omega t}$
$\sum_j (m_{jk}(-\omega^2) A_j e^{i\omega t} + V_{jk} A_j e^{i\omega t}) = 0$
$\Rightarrow \sum_j (-\omega^2 m_{jk} + V_{jk}) A_j = 0$ (f Gleichungen für f Unbekannte) $\Rightarrow$ lineares homogenes Gleichungssystem
(nicht trivial für A_j lösbar $\Leftrightarrow det(-\omega^2 m_{jk}) = 0 \Rightarrow$ Gleichungen für ω für f Werte ω_α $(\alpha = 1, ..., f)$)
(m_{jk} und V_{jk} Eigenschaften des Körpers!)
$\Rightarrow A_j^\alpha$

Allgemeine Lösung für x_j : $x_j = \sum\limits_{\alpha=1}^{f} C_\alpha A_j^\alpha e^{i\omega_\alpha t} \Rightarrow$ Eigenfrequenzen

x_j schwingt kompliziert mit vielen Frequenzen! Nur Realteil physikalisch relevant.

Führe neue Koordinate ein: $q_\alpha = C_\alpha e^{i\omega_\alpha t} \Rightarrow x_j = \sum\limits_{\alpha=1}^{f} A_j^\alpha q_\alpha, \quad j = 1, ..., f \Rightarrow q_\alpha = q_\alpha(x_j)$ linear.

$x = Aq, \ q = A^{-1}x \Rightarrow L(x, \dot{x})$ und $L(q, \dot{q})$ gleiche Gestalt, da Transfo. linear.

$L(q, \dot{q}) = \frac{1}{2} \sum\limits_k m_0(\dot{q}\alpha^2 - \omega_\alpha^2 q_\alpha^2), \quad Q_\alpha = \sqrt{m_\alpha} q_\alpha \Rightarrow L = \frac{1}{2} \sum\limits_\alpha (\dot{Q}_\alpha^2 - \omega_\alpha^2 Q_\alpha^2)$

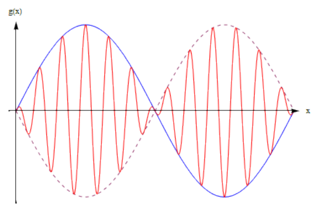
9.1 Beispiel: gekoppelte 2-dim. harm. Oszillator

$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}k(x^2 + y^2) + \alpha mxy, \quad m_{ij} = \delta_{ij} m$

$v_{11} = k, \quad v_{22} = k, \quad v_{21} = -2\alpha m \quad \Rightarrow v = \begin{pmatrix} k & -2\alpha m \\ -2\alpha m & k \end{pmatrix}, \quad m = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$

Bewegungsgleichung: $\ddot{x}^2 + \omega_0^2 x + \alpha y = 0, \quad \ddot{y}^2 + \omega_0^2 y^2 + \alpha x = 0, \quad w_0^2 = \frac{k}{m}$

Ansatz: $x_1 = A_1 e^{i\omega t}, \quad x_2 = A_2 e^{i\omega t}$



$\Rightarrow (-\omega^2 + \omega_\alpha^2)A_1 - \alpha A_2 = 0, \quad (-\omega^2 + \omega_\alpha^2)A_2 - \alpha A_1 = 0$

$\Rightarrow$ n. trivial lösbar wenn $\begin{vmatrix} \omega_0^2 - \omega^2 & -\alpha \\ -\alpha & \omega_0^2 - \omega^2 \end{vmatrix} = 0$

$\Rightarrow (\omega_0^2 - \omega^2)^2 - \alpha^2 = 0 \Rightarrow \omega_1^2 = \omega_0^2 - \alpha, \quad \omega_2^2 = \omega_0^2 + \alpha$

1\. $\omega_1 : \left.\begin{matrix} \alpha(A_1 - A_2) = 0 \\ \alpha(A_1 - A_2) = 0 \end{matrix}\right\} A_1^1 = A_2^1 \qquad \omega_2 : \left.\begin{matrix} \alpha(A_1 + A_2) = 0 \\ \alpha(A_1 + A_2) = 0 \end{matrix}\right\} A_1^2 = -A_2^2$

$\Rightarrow$ Allg. Lösung: $x = C_1 A_1^1 e^{i\omega_1 t} + C_2 A_1^2 e^{i\omega_2 t}, \quad y = C_1 A_2^1 e^{i\omega_1 t} + C_2 A_2^2 e^{i\omega_2 t}$

$\Rightarrow A_i^2 \alpha$ eingesetzt: $x = C_1 A_1^1 e^{i\omega_1 t} + C_2 A_1^2 e^{i\omega_2 t}, \quad y = C_1 A_1^1 e^{i\omega_1 t} - C_2 A_1^2 e^{i\omega_2 t}$

Schwache Kopplung: $\alpha << \omega_0^2 \Rightarrow \omega_{1,2}^2 = \omega_0^2 \mp \alpha$

$\Rightarrow \omega_{1,2} = \sqrt{\omega_0^2 \mp \alpha} = \omega_0 \sqrt{1 \mp \frac{\alpha}{w_0^2}} \approx w_0(1 \mp \frac{\alpha}{2\omega_0^2}) = w_0 \pm \frac{\alpha}{2\omega_0^2}$

$\Rightarrow x = e^{i\omega_0 t}(C_1 A_1^1 e^{-i\frac{\alpha}{2\omega_0}t} + C_2 A_1^2 e^{i\frac{\alpha}{2\omega_0}t}), \quad y = ...$

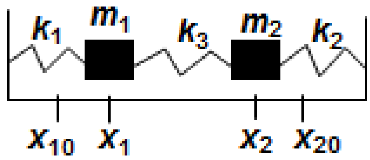
Anfangsbedingungen so, dass $C_1 A_1^2 = C_2 A_1^2 \quad (y(t=0) = 0) \Rightarrow x(t) \sim \cos(\frac{\alpha}{2\omega_0}t)e^{i\omega_0 t}$

Sei nun $\frac{\alpha}{2\omega_0} << \omega_0, \quad q_1 = B_1 e^{i\omega_1 t}, \quad q_2 = B_2 e^{i\omega_2 t}, \quad B_1 = C_1 A_1^1, \quad B_2 = C_2 A_1^2$

$x = q_1 + q_2, \quad y = q_1 - q_2 \Rightarrow q_1 = \frac{1}{2}(x + y), \quad q_2 = \frac{1}{2}(x - y)$

Schwingungen sind ungekoppelt und führen Normal-Koordinaten-Schwingungen aus.

9.2 Beispiel: gek. Oszillator mit 2 Massen und 3 Federn.



$L = \frac{m_1}{2}\dot{x}_1^2 + \frac{m_2}{2}\dot{x}_2^2 - V(x_1, x_2)$

$V(x_1, x_2) = \frac{1}{2}k_1(x_1 - x_{10})^2 + \frac{1}{2}k_2(x_2 - x_{20})^2 + \frac{1}{2}k_3((x_1 - x_{10}) - (x_2 - x_{20}))^2$

$m_1\ddot{x}_1 + k_1(x_1 - x_{10}) + k_3((x_1 - x_{10}) - (x_2 - x_{20})) = 0$

$m_2\ddot{x}_2 + k_2(x_2 - x_{20}) - k_3((x_1 - x_{10}) - (x_2 - x_{20})) = 0$

$q_1 = x_1 - x_{10}, \quad q_2 = x_2 - x_{20}$

$\Rightarrow m_1\ddot{q}_1 + k_1 q_1 + k_3(q_1 - q_2) = 0, \quad m_2\ddot{q}_2 + k_2 q_2 - k_3(q_1 - q_2) = 0 \Rightarrow m_1\ddot{q}_1 + m_2\ddot{q}_2 + k_1 q_1 + k_2 q_2 = 0$

Sei $m_1 = m_2 = m, \quad k_1 = k_2 = k \Rightarrow m(\ddot{q}_1 + \ddot{q}_2) + k(q_1 + a_2) = 0$

$X = q_1 + q_2 \Rightarrow m\ddot{X} + kX = 0 \Rightarrow X(t) = Ae^{i\omega_0 t}, \quad \omega_0 = \sqrt{\frac{k}{m}}$

$m(\ddot{q}_1 + \ddot{q}_2) + k(q_1 - q_2) + 2k_3(q_1 - q_2) = 0 \Rightarrow Y = q_1 - q_2 \Rightarrow m\ddot{Y} + (k + 2k_3)Y = 0$

$\Rightarrow Y(t) = Be^{i\omega_1 t}, \quad \omega_1 = \sqrt{\frac{k+2k_3}{m}}$

X und Y sind Normalmoden mit der Normalfrequenz $\omega_0,\ \omega_1$
$X = q_1(Ae^{i\omega_0 t} + Be^{i\omega_1 t}) \cdot \frac{1}{2}, \qquad Y = q_2(Ae^{i\omega_0 t} - Be^{i\omega_1 t}) \cdot \frac{1}{2}$
$q_1 = \frac{A}{2}e^{i\omega_0 t}(1 + \frac{B}{A}e^{i(\omega 1 - \omega_0)t}), \qquad q_2 = \frac{A}{2}e^{i\omega_0 t}(1 - \frac{B}{A}e^{i(\omega 1 - \omega_0)t}) \Rightarrow$ Schwebung!
$\omega_1 - \omega_0 = \omega_0\sqrt{1 + \frac{2k_3}{k}} - \omega_0 \approx \omega_0(1 + \frac{k_3}{k} + ...) - \omega_0$ für $k_3/k << 1$
$\Rightarrow \omega_1 - \omega_0 = \frac{k_3}{k}\omega_0 << \omega_0$

10 Bewegungen in beschleunigten Bezugssystemen

10.1 a) geradlinige Bewegung (keine Rotation)

$\vec{v} = \vec{V} + \vec{v}', \vec{r} = \vec{R} + \vec{r}'$ (U: Potential, V: Geschw. des Systems, v':Teilchengeschw. in System, v:Teilchengeschw.
$\Rightarrow L = \frac{m}{2}\vec{v}^2 - U = \frac{m}{2(\vec{V}+\vec{v}')^2 - U =}\frac{m}{2}\vec{V}^2 + m\vec{V}\vec{v}' + \frac{m}{2}\vec{v}'^2 - U$
$\vec{V} \cdot \vec{v}' = \vec{V} \cdot \frac{d\vec{r}'}{dt} = \frac{d}{dt}(\vec{V} - \vec{r}') - \vec{r}' \cdot \frac{d\vec{V}}{dt}$
Hamiltonsches Prinzip: $\delta \int\limits_{t_0}^{t_1} Ldt = 0, \quad L \to L' = L + \frac{dF}{dt}$
$\delta \int Ldt \Rightarrow \delta \int L'dt = \delta \int\limits_{t_0}^{t_1} Ldt + \int \frac{dF}{dt}dt = \delta \int\limits_{t_0}^{t_1} Ldt + \underbrace{\delta(F(t_1) - F(t_0))}_{0} \Rightarrow \delta \int L'dt = \int Ldt$
$\Rightarrow$ L ist bestimmt bis auf totale zeitliche Ableitung.
äquivalent $L = \frac{m}{2}V^2 + \underbrace{\frac{m}{2}v'^2 - m\vec{r}' - \frac{d\vec{V}}{dt} - U}_{L' bew.Sys.}$

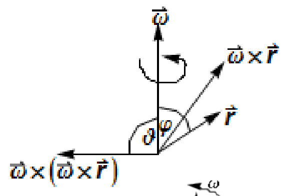
10.2 b) allgemeine Bewegungen und Rotationen

$\vec{v} = \vec{V} + (\vec{\omega} \times \vec{r}') - \vec{v}'$
Eingesetzt in L': $L' = \frac{m}{2}\vec{v}'^2 - m\vec{r}''\frac{d\vec{V}}{dt} - U$ ($\vec{v}''$: Geschw. in Rotation)
$\vec{v}' = (\vec{\omega} \times \vec{r}'') - \vec{v}''$
$\Rightarrow \frac{m}{2}(\vec{v}'' + (\vec{\omega} \times \vec{r}''))^2 - m\vec{r}''\frac{d\vec{V}}{dt} - U \Rightarrow \frac{m}{2}\vec{v}''^2 + \frac{m}{2}(\vec{\omega} \times \vec{r}'')^2 + m\vec{v}''(\vec{\omega} \times \vec{r}'') - m\vec{r}''\frac{d\vec{V}}{dt} - U$

10.3 Bewegungsgleichungen

$\frac{\partial L}{\partial}v_i = mv_i + m(\vec{\omega} \times \vec{r})_i, \quad \frac{\partial L}{\partial x_i} =?$
$\frac{\partial}{\partial x_i}(\vec{\omega} \times \vec{r}) = 2(\vec{\omega} \times \vec{r})\frac{\partial}{\partial x_i}(\vec{\omega} \times \vec{r})$
$\frac{\partial}{\partial x_i}(\vec{\omega} \times \vec{r})_k = \frac{\partial}{\partial x_i}\varepsilon_{klm}\omega_l x_m = \varepsilon_{kli\omega_l} \Rightarrow (\vec{\omega} \times \vec{r})_k \frac{\partial}{\partial x_i}(\vec{\omega} \times \vec{r})_k = (\vec{\omega} \times \vec{r})_k \varepsilon_{kli}\omega_l$
$= ((\vec{\omega} \times \vec{r}) \times \vec{\omega})_i$
$\frac{\partial}{\partial x_i}\vec{v} \cdot (\vec{\omega} \times \vec{r}) = \frac{\partial}{\partial x_i}v_k(\vec{\omega} \times \vec{r})_k = v_k\frac{\partial}{\partial x_i}(\vec{\omega} \times \vec{r})_k = v_k\varepsilon_{klm}\omega_l\frac{\partial}{\partial x_i}x_m = v_k\varepsilon_{kli}\omega_l = (\vec{v} \times \vec{\omega})_i$
$\Rightarrow \frac{d}{dt}(\frac{\partial L}{\partial v_i}) - \frac{\partial L}{\partial x_i} = \frac{d}{dt}(mv_i + m(\vec{\omega} \times \vec{r})_i) - m((\vec{\omega} \times \vec{r}) \times \vec{\omega})_i - m(\vec{v} \times \vec{\omega})_i + m\frac{dV}{dt} + \frac{\partial U}{\partial x_i}$
$\Rightarrow m\frac{dv_i}{dt} = -\frac{\partial U}{\partial x_i}\underbrace{-m\frac{dV}{dt} + m(\vec{r} \times \dot{\vec{\omega}})_i + 2m(\vec{v} \times \vec{\omega})_i + m((\vec{\omega} \times \vec{r}) \times \vec{\omega})_i}_{Scheinkraft}$

10.4 Scheinkräfte

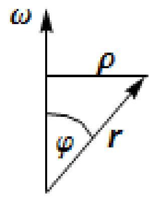


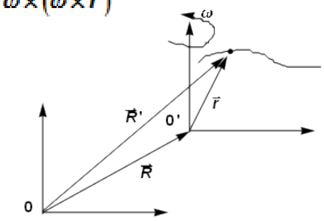
$m\frac{d\vec v}{dt} = -\vec\nabla U - m\frac{d\vec V}{dt} + m(\vec r\times\dot{\vec\omega}) + 2m(\vec v\times\vec\omega) - m(\vec\omega\times(\vec\omega\times\vec r))$

1) $m\vec r\times\dot{\vec\omega} \Rightarrow$ Änderung der Drehgeschwindigkeit

2) $2m\vec v\times\vec\omega \Rightarrow$ Corioles-Kraft

3) $-m(\vec\omega\times(\vec\omega\times\vec r)) \Rightarrow$ Zentrifugalkraft
   Radial nach außen $\perp\vec\omega$
   $|\vec\omega\times(\vec\omega\times\vec r))| = \omega|\vec\omega\times\vec r|\sin(\vartheta) = \omega\omega r\sin(\varphi) = \omega^2\varrho$



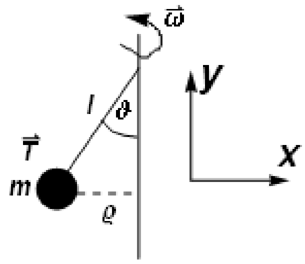


$\vec R' = \vec R + \vec r,\quad m\ddot{\vec r} = -\vec\nabla U - m\ddot{\vec R} + m(\vec r\times\dot{\vec\omega}) + \underbrace{2m(\vec r\times\vec\omega)}_{Corioleskr.} - \underbrace{m(\vec\omega\times(\vec\omega\times\vec r))}_{Zentrifugalkr.}$

10.5 Energieerhaltung in beiden Bezugssystemen

$\frac{\partial L}{\partial t} = 0 \Rightarrow E = p\dot q - L$ erhalten
Energie im bewegten System: $L' = \frac{m}{2}v^2 + m\vec v(\vec\omega\times\vec r) + \frac{m}{2}(\vec\omega\times\vec r)^2 - U \qquad (\dot{\vec\omega} = \ddot{\vec R} = 0)$
$p = \frac{\partial L'}{\partial\dot q} \Rightarrow p_i = \frac{\partial L'}{\partial v_i} = mv_i + m(\vec\omega\times\vec r)_i = m\dot R'_i = p_0$ (general. Impuls im System 0)
$E' = p_i\dot q_i - L' = m(v_i + (\vec\omega\times\vec r)_i)\dot x_i - \frac{m}{2}\vec v^2 - m\vec r - m\vec v(\vec\omega\times\vec r) - \frac{m}{2}(\vec\omega\times\vec r)^2 + U$
$= \frac{m}{2}v^2 - \frac{m}{2}(\vec\omega\times\vec r)^2 + U$
(Für R=0) $(\vec v_0 = \dot{\vec R}' = \vec v + \vec\omega\times\vec r)$
$E' = \frac{m}{2}(\vec v_0 - \vec\omega\times\vec r)^2 - \frac{m}{2}(\vec\omega\times\vec r)^2 + U = \frac{m}{2}v_0^2 - m\vec v_0(\vec\omega\times\vec r) + U = E \;\underbrace{-m\vec v_0(\vec\omega\times\vec r)}_{=-\vec\omega(\vec r\times m\vec v_0)=-\vec\omega\cdot\vec l}$

10.6 Beispiel: Gewicht an Faden rotiert um z-Achse

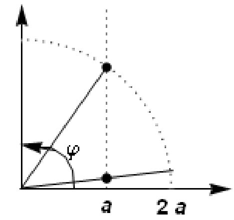


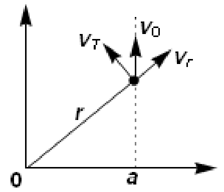
a) Lab Sys: $\vec\omega = (0,\omega)$
   Kräfte:
   1) Gewichtskraft: $(0,-mg)$
   2) Seilspannung: $(T\sin(\vartheta), T\cos(\vartheta))$
   3) Zentripetalkraft: $(m\omega^2\varrho, 0)$

   $\vec F_z = \vec T + \vec G$ (Bedingung für stabile Bewegung)
   $m\omega^2\varrho = T\sin(\vartheta),\quad 0 = T\cos(\vartheta) - mg$
   $m\omega^2 l\sin(\vartheta) = T\sin(\vartheta) \Rightarrow T = mw^2 l = \frac{mg}{\cos(\vartheta)}$ $(\vartheta\neq 0)$, $\cos(\vartheta) = \frac{g}{l\omega^2}$

b) Bewegtes System: Gleichgewbew.: $F_{total} = 0 = \vec G + \vec T + \underbrace{\vec F_z}_{Zentrifugalkr.}$

10.7 Beispiel: Masse in drehendem Rohr





keine Gravitation! $v_0 = \omega_0 a \overset{!}{=} const$

Austrittswinkel: $\varphi_a = \frac{\pi}{3} \quad (\cos(\varphi_a) = \frac{1}{2})$

$\frac{a}{r} = \cos(\varphi) \Rightarrow r(\varphi) = \frac{a}{\cos(\varphi)}, \quad v_t = \omega r = v_0 \cos(\varphi) = \omega_0 a \cos(\varphi)$

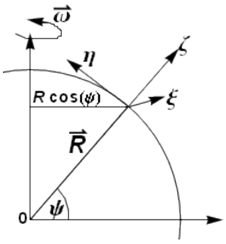
$\Rightarrow \omega_a = \omega_0 a \cos^2(\varphi) \Rightarrow \dot{\varphi} = \omega_0 \cos^2(\varphi) \Rightarrow \frac{d\varphi}{\cos^2(\varphi)} = \omega_0 dt$

Anfangebedingungen: $t = 0, \quad \varphi = 0 \ \Rightarrow \int\limits_0^{\varphi} \frac{d\varphi'}{\cos^2(\varphi')} = \omega_0 t = \tan(\varphi)$

$\omega(t) = \omega_0 \cos^2(\varphi) = \frac{\omega_0}{1+1\tan^2(\varphi)} = \frac{\omega_0}{1+(\omega_0 t)^2}$ für freie Bewegung!

Austrittszeit $= \frac{d}{v_0} = \frac{d}{\omega_0 a} = \frac{\sqrt{3}a}{\omega_0 a} = \frac{\sqrt{3}}{\omega_0}$

10.8 Bewegung auf Erde



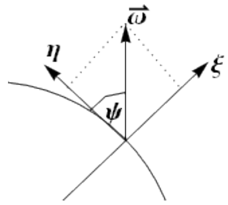
$m\ddot{\vec{r}} = -\vec{\nabla} U + m(-\ddot{\vec{R}} - \dot{\vec{\omega}} \times \vec{r} - \vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2\vec{\omega} \times \vec{v})$

$R \approx 6,4 \cdot 10^3 km (Erdradius), \quad \dot{\vec{R}} = \vec{\omega} \times \vec{R}, \quad \ddot{\vec{R}} = \vec{\omega} \times \dot{\vec{R}} = \vec{\omega} \times (\vec{\omega} \times \vec{R}), \quad \vec{\omega} = const$

Komponenten von ω im erdfesten System $(\xi, \eta, \zeta) \Rightarrow \vec{\omega} = \omega(0, \cos(\psi), \sin(\psi))$

$\vec{R} = (0, 0, R), \quad \ddot{\vec{R}} = \omega^2 R \cos(\psi) \cdot (0, \sin(\psi), -\cos(\psi))$

Annahme: $r << R \Rightarrow \vec{\omega} \times (\vec{\omega} \times \vec{r})$ vernachlässigbar. (r: Abstand von Erdoberfl.)



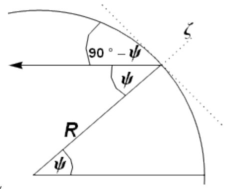
Bewegungsgleichungen:

$m\ddot{\xi} = F_\xi + m(-2\vec{\omega} \times \vec{v})_\xi = F_\xi + m(-2\omega(\cos(\psi)\dot{\zeta} - \sin(\psi)\dot{\eta}))$

$m\ddot{\eta} = F_\eta + m(-\omega^2 R \cos(\psi)\sin(\psi) - 2\omega \sin(\psi)\dot{\xi})$

$m\ddot{\zeta} = F_\zeta + m(\omega^2 R \cos^2(\psi) + 2\omega\cos(\psi)\dot{\xi})$

Annahme: $\vec{F} = (0, 0, -mG), \ \vec{v} = 0$ (ruhende Masse)



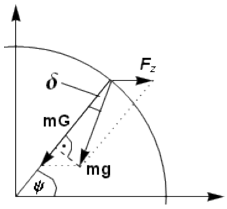
$m\ddot{\xi} = 0, \quad m\ddot{\eta} = -m\omega^2 R\cos(\psi)\sin(\psi), \quad m\ddot{\zeta} = -mG + m\omega^2 R \cos^2(\psi)$

$\Rightarrow g = \sqrt{\ddot{\eta}^2 + \ddot{\zeta}^2} = \sqrt{(\omega^2 R)^2 \cos^2(\psi)\sin^2(\psi) + (-G + \omega^2 R\cos^2(\psi))^2}$

$\omega^2 R \approx (\frac{2\pi}{24h})^2 \cdot 6.4 \cdot 10^3 km = (\frac{1}{4})^2 \cdot 6,4 \cdot 10^3 \frac{km}{h^2} \approx 4 \cdot 10^2 \frac{km}{h^2} = 4 \cdot \frac{10^5}{(60^2)^2}\frac{m}{s^2} \approx 3 \cdot 10^{-2}\frac{m}{s^2}$

$\omega^2 R << G \Rightarrow g = \sqrt{G^2 - 2\omega^2 RG\cos^2(\psi)} \approx G - \omega^2 R\cos^2(\psi) \approx (9,81 - 0,03\cos^2(\psi))\frac{m}{s^2}$

$\sin(\delta) = \frac{|F_{z\eta}|}{mg} = \frac{|\omega^2 R\cos(\psi)\sin(\psi)|}{|G - \omega^2 R\cos^2(\psi)|} \approx 0,003\sin(2\psi)$



11 Fall und Wurf auf der Erde

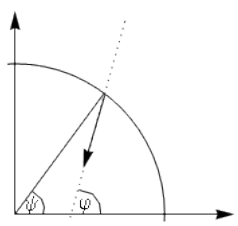
$\vec{v} \neq \vec{0}$

$\vec{F} = (0, 0, -mG) = (0, 0, -mg)$ (neues KS so, dass z-Achse in Ri. der eff. Grav.) (x,y,z)

$m\ddot{x} = F_x + m(-2\omega(\dot{z}\cos(\psi) - \dot{y}\sin(\psi)))$

$m\ddot{y} = F_y + m(-2\omega\dot{x}\sin(\psi)), \quad m\ddot{z} = F_z + m(2\omega\dot{x}\cos(\psi)) \quad (\psi \approx \varphi)$

$$\Rightarrow \left| \begin{array}{ll} m\ddot{x} & = -2m\omega(\dot{z}\cos(\psi) - \dot{y}\sin(\psi)) \\ m\ddot{y} & = -2m\omega(\dot{x}\sin(\psi)) \\ m\ddot{z} & = -mg + 2m\omega\dot{x}\cos(\psi) \end{array} \right|$$



11.1 Beispiel: Freier Fall

$\dot{x}, \dot{y} << \dot{z} \qquad \Rightarrow m\ddot{x} \approx -2m\omega\dot{z}\cos(\psi), \quad m\ddot{y} \approx 0, \quad m\ddot{z} \approx -mg$

$x_0 = y_0 = 0, \quad z_0 = h \qquad \Rightarrow z(t) = h - \frac{1}{2}gt^2 \Rightarrow \ddot{x} = -2\omega\cos(\psi)(-gt)$

$\Rightarrow \ x(t) = \frac{1}{3}\omega g t^3 \cos(\psi)$ (Richtung nach Osten!)

$z(t_f) = 0 \Rightarrow t_f = \sqrt{\frac{2h}{g}} \Rightarrow x(t_f) = \omega g\cos(\psi) \cdot \frac{1}{3}(\frac{2h}{g})^{\frac{3}{2}}$

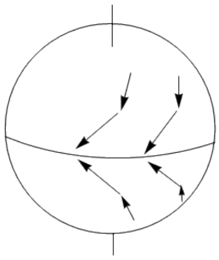
$\omega = 7.3 \cdot 10^{-5}\frac{1}{s}, \quad \cos(\psi) \approx \frac{1}{2}, \quad h \approx 100m \quad \Rightarrow x(t_f) = 7.3 \cdot 10^{-5} \cdot 9.81 \cdot \frac{1}{2} \cdot \frac{1}{3}(\frac{2 \cdot 100}{9.81})^{\frac{3}{2}} = 1.1cm$

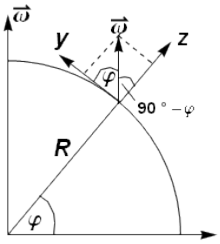
1. Iteration: $\ddot{x} \ = -2\omega(\dot{z}\cos(\varphi)) \ = -2\omega(-gt)\cos(\varphi)$

$\ddot{y} \ = -2\omega(\dot{x}\sin(\varphi)) \ = -2\omega(2\omega\cos(\varphi)\frac{gt^2}{2}\sin(\varphi))$

$\ddot{z} \ = -g + 2\omega\cos(\varphi)\frac{gt^2}{2}$

11.2 Beispiel: senkrechter Wurf





$x(0) = y(0) = z(0) = 0; \quad \dot{x}(0) = \dot{y}(0) = 0; \quad \dot{z}(0) = v_0$

$\ddot{x} = 2\omega\cos(\varphi)\dot{z}, \quad \ddot{y} = 0, \quad \ddot{z} = -g$

$\dot{z} = -gt + v_0 \Rightarrow z(t) = -\frac{1}{2}gt^2 + v_0 t$

Zeit bis Umkehrpunkt: $t_u = \frac{v_0}{g} \Rightarrow h = z(t_u) = \frac{1}{2}\frac{v_0^2}{g}$

$\ddot{x} = -2\omega\cos(\varphi)\dot{z} = -2\omega\cos(\varphi)(v_0 - gt) \Rightarrow \dot{x} = 2\omega\cos(\varphi)(\frac{gt^2}{2} - v_0 t)$

$x(t) = 2\omega\cos(\varphi)(g\frac{1}{2}\frac{t^3}{3} - \frac{1}{2}v_0 t^2), \quad x(t_u) = \omega\cos(\varphi)(g\frac{1}{3}(\frac{v_0}{g})^3 - v_0(\frac{v_0}{g})^2) = -\frac{2}{3}\omega\cos(\varphi)\frac{v_0^3}{g^2}$

(Westablenkung!)

$x(t_f) = 2\omega\cos(\varphi)(\frac{1}{2}g\frac{1}{3}(\frac{2v_0}{g})^3 - v_0\frac{1}{2}(\frac{2v_0}{g})^2) = -\frac{4}{3}\omega\cos(\varphi)\frac{v_0^3}{g}$

11.3 Beispiel: waagrechter Wurf

$\ddot{x} = -2\omega(\dot{z}\cos(\varphi) - \dot{y}\sin(\varphi)), \quad \ddot{y} = -2\omega\dot{x}\sin(\varphi), \quad \ddot{z} = -g + 2\omega\dot{x}\cos(\varphi)$

$\dot{z} << \dot{x}, \dot{y} \Rightarrow \ddot{x} = 2\omega\sin(\varphi)\dot{y}, \quad \ddot{y} = -2\omega\sin(\varphi)\dot{x}, \quad \ddot{z} = -g + 2\omega\cos(\varphi)\dot{x}$

$2\omega\cos(\varphi) \approx 14,6 \cdot 10^{-5} \cdot \frac{1}{2}\frac{1}{s} \approx 7.3 \cdot 10^{-5}\frac{1}{s}$

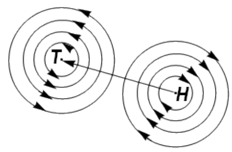
$(7.3 \cdot 10^{-5} \cdot 3 \cdot 10^2 \frac{m}{s^2} \approx 21 \cdot 10^{-3}\frac{m}{s}kg)$

$\omega_z = \omega\sin(\varphi), \quad \omega_y = \omega\cos(\varphi), \quad \ddot{x} = 2\dot{y}\omega_z, \quad \ddot{y} = -2\dot{x}\omega_z$

Bewegungsfrei (Intertialsystem), wenn dieses mit $-\omega\sin(\varphi)$ auf Erde dreht.

$\dot{y} > 0$: Ablenkung nach rechts, $\quad \dot{y} < 0$: Ablenkung nach rechts

$\dot{x} > 0$: Ablenkung nach rechts, $\quad \dot{x} < 0$: Ablenkung nach rechts



12 Hamilton'sche Mechanik

Lagrange Bewegungsgl.: $\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}}) - \frac{\partial L}{\partial q} = 0$

Kanonischer Impuls: $p = \frac{\partial L}{\partial \dot{q}}, \quad (q, \dot{q}) \Rightarrow (q, p)$

Def: Hamilton-Fu': $H(q, p, t) = p\dot{q} - L(q, \dot{q}, t)$

$L(q, \dot{q}, t) = \frac{1}{2}\sum_{ij} m_{ij}(q)\dot{q}_i\dot{q}_j - V(q, t)$, ($m_{ij}$(p) skleronome ZB)

$p_k \frac{\partial L}{\partial \dot{q}_k} = \sum_j m_{kj}(q)\dot{q}_j \Rightarrow p = p(q, \dot{q}) \Leftrightarrow \dot{q} = \dot{q}(q, p)$

$\Rightarrow H(q, p, t) = p(q, p) - L(q, (q, p), t)$

12.1 Zeitabhängigkeit von H(Variation von H)

$\partial H = \frac{\partial H}{\partial q}dq + \frac{\partial H}{\partial p}dp + \frac{\partial H}{\partial t}dt$

nach Definition: $= \frac{\partial H}{\partial q}dq + \frac{\partial H}{\partial \dot{q}}d\dot{q} + \frac{\partial H}{\partial t}dt = -\frac{\partial L}{\partial q}dq + \underbrace{(p - \frac{\partial L}{\partial \dot{q}})}_{=0}d\dot{q} + \dot{q}dp - \frac{\partial L}{\partial t}dt$

Vergleich der ∂H: $\quad \frac{\partial H}{\partial q} = -\frac{\partial L}{\partial q}; \quad \dot{q} = \frac{\partial H}{\partial p}; \quad m\frac{\partial H}{\partial t} = \frac{\partial L}{\partial t}$

$\Rightarrow \dot{q} = \frac{\partial H}{\partial p}; \quad \dot{p} = -\frac{\partial H}{\partial q}, \qquad \left(\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}\right)$ (Hamilton'sche Bewegungsgleichungen)
n konst$\Rightarrow$ 2n D-Gleichungen 1. Ordnung

12.2 Zeitliche Veränderung von H

$$\frac{dH}{dt} = \frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial p}\dot{p} + \frac{\partial H}{\partial t} = \underbrace{\frac{\partial H}{\partial q}\frac{\partial H}{\partial p} + \frac{\partial H}{\partial p}\frac{\partial H}{\partial q}}_{=0} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

$\Rightarrow$ Erhaltungssatz: Wenn $\frac{\partial H}{\partial t} = \frac{\partial L}{\partial t} = \frac{dH}{dt} = 0$, dann H konstant

$H = \sum_i p_i \dot{q}_i - L(q_i, \dot{q}_i, t) = \sum_i \left(p_i \dot{q}_i - \frac{1}{2} \sum_j m_{ij}(q)\dot{q}_i\dot{q}_j + V(q_i, t) \right)$

$p_k = \frac{\partial L}{\partial \dot{q}_k} = \sum_j m_{kj} q_j = \frac{\partial T}{\partial \dot{q}_k}$

$\sum_k p_k \dot{q}_k = \sum_k \frac{\partial T}{\partial \dot{q}_k}\dot{q}_k = \sum_{kj} m_{kj}\dot{q}_k\dot{q}_j + V(q_i, t)$

$\Rightarrow H = 2T - T - V = T + V$ (Energie für skleronone ZB)

12.3 Vorgehensweise

1) $L = T - V = L(q, \dot{q}, t)$
2) $p = \frac{\partial L}{\partial \dot{q}} \Rightarrow \dot{q} = \dot{q}(q, p)$
3) $H = p\dot{q} - L = H(q, p, t)$

12.4 Beispiel 1

1) $L = \frac{1}{2}m\dot{x}^2 - V(x) = L(x, \dot{x})$
2) $p = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \Rightarrow \dot{x} = \frac{p}{m}$
3) $H = p\dot{x} - L(x, \dot{x}) = p\frac{p}{m} - (\frac{p^2}{2m} - V) = \frac{p^2}{2m} + V = H(x, p)$

$\Rightarrow$ Bewegungsgleichung: $\dot{x} = \frac{p}{m}, \quad \dot{p} = -\frac{\partial V}{\partial x}$

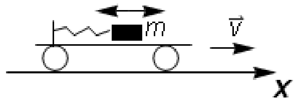
12.5 Beispiel 2

Zylinderkoord: $\varrho, z, \varphi \Rightarrow T = \frac{m}{2}(\dot{z}^2 + \dot{\varrho}^2 + \varrho^2\dot{\varphi}^2)$

1) $L = \frac{m}{2}(\dot{z}^2 + \dot{\varrho}^2 + \varrho^2\dot{\varphi}^2) - V(\varrho, z, \varphi) = L(q, \dot{q})$
2) $p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z}, \quad p_\varrho = \frac{\partial L}{\partial \dot{\varrho}} = m\dot{\varrho}, \quad p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m\varrho^2\dot{\varphi}$ (=Drehimpuls)
3) $H = p_z\dot{z} + p_\varrho\dot{\varrho} + p_\varphi\dot{\varphi} - L(\varrho, z, \varphi) = \frac{p_z^2}{m} + \frac{p_\varrho^2}{m} + \frac{p_\varphi^2}{m\varrho^2} - \frac{p_z^2}{2m} - \frac{p_\varrho^2}{2m} - \varrho^2(\frac{p_\varphi}{m\varrho^2})^2\frac{m}{2}$
$= \frac{p_z^2}{2m} + \frac{p_\varrho^2}{2m} + \frac{p_\varphi^2}{2m\varrho^2} + V$
$\Rightarrow \dot{q} = \frac{\partial H}{\partial q}, \quad \dot{p} = -\frac{\partial H}{\partial q}$
$\Rightarrow \dot{p}_z = -\frac{\partial V}{\partial z}; \quad \dot{p}_\varrho = -\frac{\partial V}{\partial \varrho} + \frac{p_\varrho^2}{m\varrho^3}; \quad \dot{p}_\varphi = -\frac{\partial V}{\partial \varphi}$

12.6 Beispiel 3: harm osz. auf Wagen mit v=const

Ein Harmonischer Oszillator bedindet sich auf einem Wagen, der sich mit konstanter Geschwindigkeit v_0 bewegt. Bewegungen seien 1-Dimensional.



1) Labor: $H = \frac{p^2}{2m} + \frac{1}{2}k(x - v_0 t)^2 = H(x, p, t)$
   $\dot{p} = -\frac{\partial H}{\partial x} = -k(x - v_0 t), \quad \dot{x} = \frac{p}{m}$
   Substitution: $x' = x - v_0 t \;\Rightarrow\; \ddot{x}'m = -kx' \;\Rightarrow$ harm. Schwingung
   $\Rightarrow \frac{\partial H}{\partial t} \neq 0 \Rightarrow$ Energieerhaltung für konst. Geschw.
   Lagrange-Funktion: $L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}h(x - v_0 t)^2$

2) Bewegtes System: $x' = x - v_0 t \Rightarrow \dot{x}' = \dot{x} - v_0$ (Galilei)
   $\Rightarrow L = \frac{1}{2}m(\dot{x}' + v_0)^2 - \frac{1}{2}hx'^2 = \frac{m}{2}\dot{x}'^2 + mv_0\dot{x}' + \frac{m}{2}v_0^2 - \frac{1}{2}kx'^2$
   $p' = (\frac{\partial L}{\partial \dot{x}'})m\dot{x}' + mv_0 \Rightarrow \dot{x}' = \frac{1}{m}(p' - mv_0) \Rightarrow H = p'\dot{x}' - L(x', \dot{x}')$
   $= p' \cdot \frac{1}{m}(p' - m_0 v) - \frac{m}{2}(\frac{1}{m}(p' - mv_0))^2 - mv_0\frac{1}{m}(p' - mv_0) - \frac{m}{2}v_0^2 + \frac{1}{2}kx'^2$
   $H(x', p') = \frac{\dot{p}^2}{2m} + \frac{1}{2}kx'^2 - v_0 p' \Rightarrow \frac{\partial H}{\partial t} = 0 \Rightarrow$ H=const=Energie
   $H = (\frac{p' - mv_0}{2m})^2 + \frac{1}{2}kx'^2 - \frac{1}{2}mv_0^2 \Rightarrow \tilde{H} = (\frac{p' - mv_0}{2m})^2 + \frac{1}{2}kx'^2 \Rightarrow$ gleiche Bewegungsgl.
   $\frac{\partial \tilde{H}}{\partial t} = 0 \Rightarrow \tilde{H}$zeitl. konst$\neq$ Energie!

12.7 Kurzwiederholung

L=T-V

1) $\frac{d}{dt}\frac{\partial}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$
   $\Rightarrow$ n Gleichungen 2 Ordnung für n Variabeln und 2n Randbedingungen
2) konjugierte Impulse $p_i = \frac{\partial L}{\partial \dot{q}_i}$
3) $H(q_i, \dot{q}_i, p, t) = \sum p_i \dot{q}_i - L(q_i, \dot{q}_i, t)$
4) $\dot{q}_i = \dot{q}_i(q, p, t)$
5) $H(q_i, p, t)$

$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \Rightarrow$ 2n Gleichungen 1. Ordnung

12.8 Zyklische Variablen

q_n „zyklisch", falls nicht explizit in L
$L = L(q_1, ..., q_{n-1}, \dot{q}_1, .., \dot{q}_n, t) \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_n} - \cancel{\frac{\partial L}{\partial q_n}} = 0 \Rightarrow \frac{d}{dt}p_n = 0 \Rightarrow p_n = const$
$H = H(q_1, ..., q_{n-1}, \dot{q}_i, ..., \dot{q}_n = \beta, t), \beta = const \Rightarrow n - 1$ Freiheitsgrade

12.9 Poisson-Klammer

bel. Fu. $f(q_i, p_i, t)$
$\frac{df}{dt} = \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial q_i}{\partial t} + \frac{\partial f}{\partial p_i}\frac{\partial p_i}{\partial t}\right) + \frac{\partial f}{\partial t} = \sum \left(\frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p_i}\dot{p}_i\right) + \frac{\partial f}{\partial t} = \sum \left(\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} + \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i}\right) + \frac{\partial f}{\partial t}$
$\{g, f\} := \sum_i \left(\frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i} - \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q_i}\right)$ (Poisson-Klammer), $\quad \{g, f\} = -\{f, g\}$
$\Rightarrow \frac{df}{dt} = \frac{\partial f}{\partial t} + \{f, H\} \Rightarrow$ f Erhaltungsgröße $\Leftrightarrow \frac{df}{dt} = \{f, H\}$

- $f = H \Rightarrow \frac{dH}{dt} = \frac{\partial H}{\partial t} + \{H, H\} = \frac{\partial H}{\partial t}$
- $g = q_j \Rightarrow \{q_j, f\} = \sum\limits_i \left(\frac{\partial q_j}{\partial q_i}\frac{\partial f}{\partial p_i} - \frac{\partial q_j}{\partial p_i}\frac{\partial f}{\partial q_i} \right) = \sum\limits_i \delta_{ij} \frac{\partial f}{\partial p_i} = \frac{\partial f}{\partial p_j}$
- $g = p_j \Rightarrow \{p_j f\} = \sum\limits_i \left(\frac{\partial p_j}{\partial q_i}\frac{\partial f}{\partial p_i} - \frac{\partial p_j}{\partial p_i}\frac{\partial f}{\partial q_i} \right) = -\frac{\partial f}{\partial q_j}$
  $\dot{q}_i = \frac{\partial H}{\partial p_i} = \{q_i, H\}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} = \{p, H\}$
- $\{p_l, p_k\} = \sum \left(\frac{\partial p_l}{\partial q_i}\frac{\partial p_k}{\partial p_i} - \frac{\partial p_l}{\partial p_i}\frac{\partial p_k}{\partial q_i} \right) = 0$
- $\{q_l, p_k\} = \sum \left(\frac{\partial q_l}{\partial q_i}\frac{\partial p_k}{\partial p_i} - \frac{\partial q_l}{\partial p_i}\frac{\partial p_k}{\partial q_i} \right) = \sum\limits_i \delta_{il}\delta_{ik} = \delta_{lk}$
  (in Quantenmech. Kommutator, Heisenberg. Beweggl.)

12.10 kanonische Transformationen

Zyklische Variabeln vereinfachen H $\Rightarrow$ finde Koordinatensatz mit möglichst vielen von ihnen. Beispiel: $q_1 = x,\ q_2 = y; \quad q_1 = r,\ q_2 = \theta;$ (bei Zentralkräfte θ zyklisch.

1) Punkttransformation: $Q_i = Q_i(q_1, ..., q_n, t)$
2) in Hamilton'sche Bewegungsgleichung: $Q_i = Q_i(q, p, t), \quad P_i = P_i(q, p, t)$
   (2n Gleichungen)

Funktionstranf. von Phasenraum „kanonische Transformation“, wenn $\dot{q} = \frac{\partial H}{\partial p},\ \dot{p} = -\frac{\partial H}{\partial q}$
$\Rightarrow \dot{Q} = \frac{\partial K}{\partial p}, \quad \dot{P} = -\frac{\partial K}{\partial Q}, \quad K = K(P, Q, t), \quad H = H(p, q, t)$

12.11 Hamilton'sches Prinzip

$\delta \int\limits_{t_1}^{t_2} L dt = \delta \int\limits_{t_1}^{t_2} (\sum\limits_i p_q \dot{q}_i - H) dt = 0$. Außerdem $\delta \int\limits_{t_1}^{t_2} (\sum P_i \dot{Q}_i - K) dt = 0$
Also $\sum p_i \dot{q}_i - H = \sum P_i \dot{Q}_i - K + \frac{d\Phi}{dt}, \quad \Phi = \Phi(q_i, p_i, Q_i, P_i)$
(4n Variabeln, 2n Gleichungen)
Annahme: $\Phi = \Phi(q_i, Q_i) \Rightarrow \frac{d\Phi}{dt} = \sum p \dot{q}_i - H + K - \sum P_i \dot{Q}_i$
Andererseits: $\frac{d\Phi}{dt} = \sum \frac{\partial \Phi}{\partial q_i} \dot{q}_i + \sum \frac{\partial \Phi}{\partial Q_i} \dot{Q}_i + \frac{\partial \Phi}{\partial t}$
Vergleich: $p_i = \frac{\partial \Phi}{\partial q_i}, \quad P_i = -\frac{\partial I}{\partial Q_i}, \quad K = H + \frac{\partial \Phi}{\partial t}$

1) $p_i = p_i(q_i, Q_i, t) \to Q_i = Q_i(p_i, q_i, t)$
2) $P_i = P_i(q_i, Q_i, t) = P_i(q_i, Q_i(p_i, q_i, t), t) = P_i(q_i, p_i, t)$

Andere Möglichkeiten: $\Phi(q, Q, t), \quad \Phi(p, P, t), \quad \Phi(p, Q, t), \quad \Phi(q, P, t)$
Beispiel: $\Phi = \sum\limits_k q_k Q_k, \quad p_i = \frac{\partial \Phi}{\partial q_i} = Q_i, \quad P_i = -\frac{\partial \Phi}{\partial Q_i} = -q_i$
$\Rightarrow$ Bewegungsgl.: $\dot{p}_i = \dot{Q}_i = -\frac{\partial H}{\partial q} = \frac{\partial H}{\partial p}, \quad \dot{q}_i = -\dot{P}_i = \frac{\partial H}{\partial p} = \frac{\partial H}{\partial Q}$
Prüfung ob $(q, p) \to (Q, P)$ kanonisch:

a) direkte Tranf. d. Bewegl. prüfe ob wieder Hamilton'sche Bewgl.
b) Prüfe ob es Erzeugende gibt.

$H = \sum p_i \dot{q}_i - L, \quad K = \sum P_i \dot{Q}_i - L$. Es muss gelten: $\frac{d\Phi}{dt} = \sum p_i \dot{q}_i - H - \sum P_i \dot{Q}_i + K$
$d\Phi = \sum \frac{\partial \Phi}{\partial q} dq + \sum \frac{\partial \Phi}{\partial Q_i} dQ_i + \frac{\partial \Phi}{\partial t} dt = \sum p_i dq_i - \sum P_i dQ_i + (K - t) dt$
$\Rightarrow$ E hängt nicht explizit von t ab, $\partial \Phi = \sum p_i dq_i - \sum P_i dQ_i$
$\sum p_i \dot{q}_i - H = \sum P_i \dot{Q}_i - K + \frac{\partial \Phi_1}{\partial t} + \sum \frac{\partial \Phi_1}{\partial q_i} \dot{q}_i + \sum \frac{\partial \Phi_1}{\partial Q_i} \dot{Q}_i$

$\Rightarrow P_i = \frac{\partial\Phi_1}{\partial q_i}, \quad P_i = -\frac{\partial\Phi_1}{\partial Q_i}, \quad K = H + \frac{\partial\Phi_1}{\partial t}$
$\sum p_i\dot{q}_i - H = \cancel{\sum P_i\dot{Q}_i} - K + \frac{\partial\Phi_2}{\partial t} + \sum \frac{\partial\Phi_2}{\partial q_i}\dot{q}_i \sum \frac{\partial\Phi_2}{\partial P}\dot{P} - \cancel{\sum \dot{Q}_i P} - \sum Q\dot{R} \Rightarrow P_i = \frac{\partial\Phi_2}{\partial q_2}, \; Q_i = \frac{\partial\Phi_2}{\partial p_i}, \; K = H + \frac{\partial\Phi_2}{\partial t}$
$\Phi = \Phi_3(P,Q,T) + qp, \quad \Phi = \Phi_4(p,P,t) + qp - QP \;\Rightarrow$ Legendre Transformation

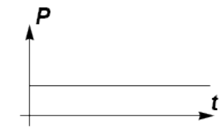
12.12 Beispiele

Beispiel 1:

$\Phi = \sum q_i Q_i = \Phi_1 : \quad p_i = \frac{\partial\Phi_1}{\partial q_i} = Q_i, \; P_i = \frac{\partial\Phi_1}{\partial Q_i} = q_i, \; K = H$
Alte Begriffe „Impuls“ und „Koordinate“ sind nun hinfällig.

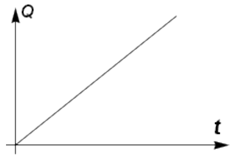
Beispiel 2:

$\Phi = \sum qP = \Phi_2 : \quad p_i = \frac{\partial\Phi_2}{\partial q_i} = P_i, \; Q_i = \frac{\partial Phi_2}{\partial p_i} = q_i \Rightarrow$ identische Abb.

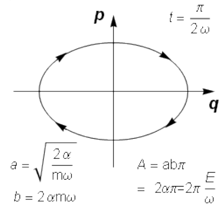


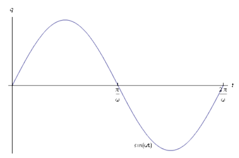
Beispiel 3 (harmonischer Oszillator):

$H = \frac{p^2}{2m} + \frac{hq^2}{2} = \frac{1}{2m}(p^2 + m^2\omega^2 q^2), \quad \omega^2 = \frac{k}{m}$
Überlegung: $p = f(P)\cos(Q), \; q = f(P)\frac{1}{m\omega}\sin(Q)$
$H = \frac{1}{2m}f^2(P)(\cos^2(Q) + \sin^2(Q)) = \frac{1}{2m}f(P)$, Q zyklisch
Ansatz: $\Phi(q,Q) = \frac{m\omega}{2}q^2\cot(Q), \; \cot = \frac{\cos}{\sin}$

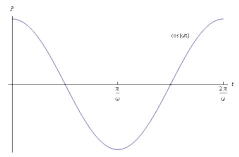


$p = \frac{\partial\Phi}{\partial q} = m\omega\cot(Q) \;\Rightarrow\; P = -\frac{\partial\Phi}{\partial Q} = -\frac{m\omega}{2}q^2\frac{\partial}{\partial Q}\frac{\cos(Q)}{\sin(Q)} = \frac{m\omega}{2}q\frac{1}{\sin(Q)}$
$\frac{\partial}{\partial Q}\frac{\cos(Q)}{\sin(Q)} = \frac{\cos'(Q)}{\sin(Q)} - \cos(Q)\frac{\sin'(Q)}{\cos^2(Q)} = -1 - \frac{\cos^2(Q)}{\sin^2(Q)} = 1 - \frac{1}{\sin^2(Q)}$





$q = \sqrt{\frac{2P}{m\omega}}\sin(Q), \; p = m\omega\sqrt{\frac{2P}{m\omega}}\sin(Q)\cdot\frac{\cos(Q)}{\sin(Q)} = \sqrt{2Pm\omega}\cos(Q)$
$K = H = \frac{1}{2m}(2Pm\omega\cos^2(Q) + m^2\omega^2\cdot\frac{2P}{m\omega}\sin^2(Q)) = \omega P$
Bewegungsgl.: $\dot{P} = \frac{\partial K}{\partial Q} = 0 \Rightarrow$ P konst$= \alpha, \; P = \frac{E}{W}$
$\dot{Q} = \frac{\partial H}{\partial P} = \frac{\partial K}{\partial P} = \omega, \; Q = \omega t + \beta, \beta$ ist konstant
wieder eingesetzt: $q = \sqrt{\frac{2\alpha}{m\omega}}\sin(\omega t + \beta), \quad p = \sqrt{2\alpha m\omega}\cos(\omega t + \beta)$



Überprüfung ob Transfo. kanonisch:
Annahme: Φ ist unbekannt, Transfo. bekannt: $q = \sqrt{\frac{2P}{m\omega}}\sin(Q), \; p = \sqrt{2Pmww}\cos(Q)$
Test: ist pdq-PdQ totales Differential?
$pdq - PdQ = p(\frac{\partial q}{\partial P}dP + \frac{\partial q}{\partial Q}dQ) - PdQ$
$= \sqrt{2Pm\omega}\cos(Q)(\sqrt{\frac{2P}{m\omega}}\cdot\frac{1}{2\sqrt{P}}\sin(Q)dP + \sqrt{\frac{2P}{m\omega}}\cos(Q)dQ) - PdQ$
$= \cos(Q)\sin(Q)dP + 2P\cos^2(Q)dQ - PdQ = \cos(Q)\sin(Q)dP + P(\underbrace{2\cos^2(Q) - 1}_{\cos^2(Q)-\sin^2(Q)})dQ$
$= (\frac{1}{2}\sin(Q))dP + P(\cos(2Q))dQ$
$d(\frac{p}{2}\sin(2Q)) = \frac{\partial}{\partial P}(\frac{p}{2}\sin(2Q))dP + \frac{\partial}{\partial Q}(\frac{p}{2}\sin(2Q))dQ = \frac{1}{2}\sin(2Q)dP + \frac{P}{2}\cdot 2\cos(2Q)dQ$
$= \frac{1}{2}\sin(2Q)dP + P\cos(2Q)dQ \Rightarrow$ totales Diff $d(\frac{p}{2}\sin(2Q))$

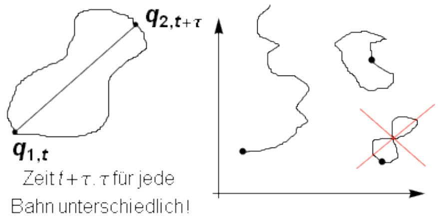
Beispiel 4:

$Q = q^a\cos(bp), \; P = p^a\sin(bp)$, a,b beliebig
$\{Q,P\}_{qp} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = aq^{a-1}\cos(bp)q^a b\cos(bp) - q^a(-\sin(bp))aq^{a-1}\sin(bp)$
$= abq^{2a-1}(\cos^2(bp) + \sin^2(bp)) = abq^{2a-1} \overset{!}{=} 0 \Rightarrow a = \frac{1}{2} < b = 2$

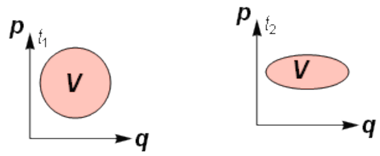
12.13 kanonische Transformation & Poisson-Klammern

$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{f, H\}$
$\{f,g\} = \frac{\partial g}{\partial q}\frac{\partial f}{\partial p} - \frac{\partial g}{\partial p}\frac{\partial f}{\partial q} = \frac{\partial g}{\partial Q}\frac{\partial f}{\partial P} - \frac{\partial g}{\partial P}\frac{\partial f}{\partial Q}$
Poisson-Klammern sind invariant unter kanonischen Transf.

12.14 zeitliche Entwicklung



$t \to t+\tau$ ist ebenfalls kanonische Transformation
$Q = q(t+\tau) = q(q(t), p(t), t), \quad P = P(t+\tau) = p(q(t), p(t), t)$
Zeige dass pdq-PdQ totales Differential
Wirkung $S(t) = \int_t^{t+\tau} L dt' = \int_t^{t+\tau} (p\dot q - H) dt'$
$\delta S = \int_t^{t+\tau} \delta L dt' = \int_t^{t+\tau} (\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot q}\delta \dot q) = \int_t^{t+\tau} (\dot p \delta q + p\delta\dot q = \int_t^{t+\tau} \frac{d}{dt'}(p\delta q) dt'$
$= p(t+\tau)\delta q(t+\tau) - p(t)\delta q(t)$
$\Rightarrow dS = p(t+\tau)dq(t+\tau) - p(t)dq(t)$
d.h. $\Phi = -S$ ist Erzeugende der kan. Transfo., die die zeitl. Entw. beschreibt.



12.15 Phasenraum

$q_1, ..., q_n; p_1, ..., p_n, \quad V = \int dq_1, ..., dq_n, dp_1, ..., dp_n$
kanonische Transfo.: $(q,p) \to (Q,P)$, $\int dQdP = |D| \int dqdp$,

$$D = \frac{\partial(Q,P)}{\partial(q,p)} = \frac{\partial(Q_1,...,Q_n,P_1,...,P_n)}{\partial(q_1,...,q_n,p_1,...,p_n)} = \begin{vmatrix} \frac{\partial Q_1}{\partial q_1} & \cdots & \frac{\partial Q_1}{\partial q_n} & \frac{\partial Q_1}{\partial p_1} & \cdots & \frac{\partial Q_1}{\partial p_n} \\ \vdots & & & & & \vdots \\ \frac{\partial P_n}{\partial q_1} & \cdots & \frac{\partial P_n}{\partial q_n} & \frac{\partial P_n}{\partial p_1} & \cdots & \frac{\partial P_n}{\partial p_n} \end{vmatrix}$$

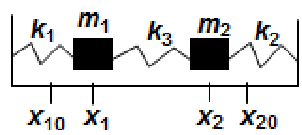
Eigenschaften:

1) $D = \frac{\partial(Q,P)}{\partial(q,p)} = \frac{\partial(Q,P)}{\partial(Q,q)}\frac{\partial(Q,q)}{\partial(q,p)}, \quad (q,p) \to (Q,q) \to (Q,P)$

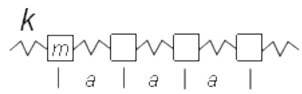
2) $\frac{\partial(Q,P)}{\partial(Q,p)} = \frac{\partial P}{\partial q}|_{Q\ const} \begin{vmatrix} \frac{\partial Q}{\partial Q} & \frac{\partial Q}{\partial q} \\ \frac{\partial P}{\partial Q} & \frac{\partial P}{\partial q} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & \frac{\partial P}{\partial q} \end{vmatrix} = \begin{vmatrix} \frac{\partial Q}{\partial q} \end{vmatrix} |_{Q\ const}$

Φ Erzeugende, $p = \frac{\partial \Phi}{\partial q}$, $P = -\frac{\partial \phi}{\partial Q}$, $D = \frac{\partial P}{\partial q}|_{Q\ const} \cdot \frac{\partial Q}{\partial p}|_{q=const} = -1$
Somit $V = \int dqdp = \int dQdP$
Das von einem Ensemble physikalischer Systeme im Phasenraum eingenommene Volumen ändert sich nicht mit der Zeit.
Also auch Dichte konstant: $\varrho = \frac{dN}{dV}$, $\frac{d\varrho}{dt} = \frac{\partial \varrho}{\partial t} + \{\varrho, H\} = 0$ (Liouvillescher Satz)

13 Kontinuumsmechanik



$L = \frac{m_1}{2}\dot q_1^2 + \frac{m_2}{2}\dot q_2^2 + \frac{1}{2}k_1 q_1^2 - \frac{1}{2}k_2 q_2^2 - \frac{1}{2}k_3(q_1 - q_2)^2, \quad q_i = x_i - x_{i0}$
$L = T - V = \frac{1}{2}\sum_{i=1}^{N} m\dot q_i^2 - \frac{1}{2}\sum_{i=1}^{N} k(q_{i+1} - q_i)^2$; q_i:Auslenkung v. Ruhelage
$L = \frac{1}{2}a\sum_{i=1}^{N}(\frac{m}{a}\dot q_i^2 - ka(\frac{q_{i+1}-q_i}{q_i})^2) = a\sum_i L_i$
$N \to \infty, a \to 0 : \frac{m}{a} \to \frac{dm}{dx} = \tau$: Materialkonst.
Def. Elastizitätsmodul $E = \frac{F}{\tilde q}, \tilde q = \frac{q_{i+1}-q_i}{a}$



$F = ka\frac{\Delta q}{a} = haq \tilde{\rightarrow} ha = E$: Materialkonst

$L \underset{a\rightarrow 0}{\longrightarrow} \int dx(\tau\dot{q}^2 - E\tilde{q}^2) = \int dx(\tau\dot{q}^2 - E(\frac{dq}{dx})^2) = \int \mathcal{L}dx,\ \mathcal{L}$:Lagrange-Dichte

Bewegungsgl.: $\frac{d}{dt}(\partial\frac{L}{\partial}\dot{q_n}) - \frac{\partial L}{\partial q_n} = 0$

$\Rightarrow \frac{m}{a}\ddot{q}_k + ka(\frac{q_k - q_{k-1}}{a^2}) - ha(\frac{q_{k+1}-q_k}{a^2}) = 0 = \frac{m}{a}\ddot{q}_k - ka(\frac{q_{k+1}-2q_k+q_{k-1}}{a^2}) \Rightarrow a \rightarrow 0 : \tau\frac{d^2q}{dt^2} - E\frac{d^2q}{dx^2} = 0$

$\Leftrightarrow (\nabla^2 - \frac{1}{v^2}\frac{d^2}{dt^2})g = 0 \ \Rightarrow\ \frac{d^2q}{dx^2 -}\frac{\tau}{E}\frac{d^2q}{dt^2} = 0$

e.g $f(x,t) = g(x - vt) \Rightarrow$ Wenn x-vt konst, dann g gleicher Wert $\Rightarrow x = const + vt$ (Welle)

$\frac{dg}{dt} = \frac{dg}{dx}(-v),\quad \frac{d^2g}{dt^2} = \frac{d^2g}{dx^2}v^2$

eingesetzt:$\frac{d^2g}{dx^2} - \frac{\tau}{E}v^2\frac{d^2g}{dx^2} = 0 \ \Rightarrow\ v^2 = \frac{E}{\tau}$

$L = \int \mathcal{L}dx, \mathcal{L} = \mathcal{L}(\frac{dq}{dx}, \frac{dq}{dt}, q, t),\quad S = \int \mathcal{L}dqdt,$ Hamilton.$\delta S = 0$

$\delta S = \int dxdt \left(\frac{\partial\mathcal{L}}{\partial(\frac{dq}{dx})}\delta\frac{dq}{dx} + \frac{\partial\mathcal{L}}{\partial(\frac{dq}{dt})}\delta\frac{dq}{dt} + \frac{\partial\mathcal{L}}{\partial x}\delta q + \frac{\partial\mathcal{L}}{\partial t}\delta t\right) \overset{!}{=} 0$

$\delta(\frac{dq}{dx}) = \frac{d}{dx}\delta q,\quad \delta(\frac{dq}{dt}) = \frac{d}{dt}\delta q$

$\delta S = \int dxdt(-\frac{d}{dx}(\frac{\partial\mathcal{L}}{\partial(\frac{dq}{dx})})\delta q - \frac{d}{dt}(\frac{\partial\mathcal{L}}{\partial(\frac{dq}{dt})})\delta q + \frac{\partial\mathcal{L}}{\partial q}\delta q) \overset{!}{=} 0$, Wenn $\frac{\partial\mathcal{L}}{\partial t} = 0$

δq beliebig $\Rightarrow$ Integrant $= 0 \Rightarrow \frac{d}{dt}(\frac{\partial\mathcal{L}}{\partial(\frac{dq}{dt})}) + \frac{d}{dx}(\frac{\partial\mathcal{L}}{\partial(\frac{dq}{dx})}) - \frac{\partial\mathcal{L}}{\partial q} = 0$ (Lagrange Gl. für q(x,t))

$\mathcal{L} = \tau(\frac{dq}{dt})^2 - E(\frac{dq}{dx})^2 \Rightarrow \frac{\partial\mathcal{L}}{\partial(\frac{dq}{dt})} = 2\tau\frac{dq}{dt},\ \frac{\partial\mathcal{L}}{\partial(\frac{dq}{dt})} = -2E\frac{dq}{dx},\quad \frac{\partial\mathcal{L}}{\partial q} = 0$

$\tau\frac{d^2q}{dt^2} - E\frac{d^2q}{dx^2} = 0$ =gleich, daher: Lagrange-Gleichung tut es!