

12 Besprechung in Analysis 2 zum Blatt 12, zum 7.7.2010

12.1

a) $U = \text{span}(v_1, \dots, v_k)$, ONB von U : $u_1, \dots, u_k$

$v_i = \sum_{j=1}^k v_{ij} u_j$ Basisdarst. $|(\det(v_{ij}))_{i,j=1\dots k}|$ $V = (v_1, \dots, v_k)$, $O = (u_1, \dots, u_k)$.

i-te Spalte: $v_i = \sum_{j=1}^k \underbrace{v_{ij}}_{\langle v_i, u_j \rangle} u_j = \sum_{j=1}^k \underbrace{\langle v_i, u_j \rangle}_{(O^T \cdot V)_{ji}} u_j = \text{i-te Spalte von } O(O^T V).$ ($V =$

$$OO^T V)$$

$$O \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} = x_1 u_1 + \dots + x_k u_k$$

b) $|\det(v_{ij})| = |\det(V^T O)| = \sqrt{\det(\underbrace{V^T O}_{k \times k}) \cdot \det(\underbrace{O^T V}_{k \times k})}$

mit Det-Mult-Satz: $\dots = \sqrt{\det(V^T \underbrace{OO^T}_V V)} = \sqrt{\det(V^T V)} = \sqrt{\det(g_{ij})}$, $g_{ij} = \langle v_i, v_j \rangle$

c) kein Problem (einsetzen), Matrix symmetrisch, determinante ausrechnen $= \sqrt{\frac{111}{2}}$.

12.2

$$\text{Vol: } \int_K 1 = \int_0^{2\pi} \pi(2 + \sin(z))^2 dz = 9\pi^2$$

$$\text{Masse} = \int_K \varrho = \int_{K \cap \{x \geq 0\}} \varrho + \int_{K \cap \{x < 0\}} \varrho = \frac{\text{vol}(K)}{2} [1 + \frac{1}{2}] = \frac{3}{4} \text{vol}(K) = \frac{27\pi^2}{4}$$

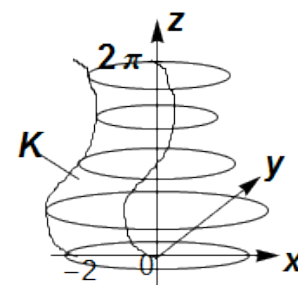
$$\int_K y = 0 \Rightarrow S_y = 0, \quad \int_K x \varrho(x, y, z) = \int_{K, x \geq 0} x \cdot 1 + \int_{K, x < 0} x \cdot \frac{1}{2}$$

$$= \frac{1}{2} \int_0^{2\pi} \left(\int_{\substack{k, x \geq 0 \\ x^2 + y^2 \leq (2 - \sin(z))^2}} x d(x, y) \right) dz = \frac{1}{2} \int_0^{2\pi} \int_{-(...) }^{+(...) } \left(\int_0^{\sqrt{(...) ^2 - y^2}} x dx \right) dy dz = \frac{1}{2} \int_{-(...) }^{+(...) } \frac{1}{2} ((...) ^2 - y^2) dy dz$$

$$\Rightarrow \dots = \frac{1}{3} \int_0^{2\pi} (z + \sin(z))^3 dz = \frac{22}{3} \pi \Rightarrow S_x = \frac{\frac{22}{3} \pi}{\frac{27\pi^2}{4}} = \frac{88}{81\pi}$$

$$\int_K z \varrho = \int_0^{2\pi} z \cdot \left(1 \cdot r(z)^2 \frac{\pi}{2} + \frac{1}{2} r(z)^2 \frac{\pi}{2} \right) dz = \frac{3}{2} \pi \underbrace{\int_0^{2\pi} z \cdot (2 + \sin(z))^2 dz}_{9\pi^2 - 8\pi \quad (?)} \Rightarrow s_z = \frac{\frac{3\pi}{2} (9\pi^2 - 8\pi)}{\frac{27\pi^2}{4}}$$

$$(\dots) := 2 + \sin(z)$$



12.3

Koordinatenlinien senkrecht, also Cramer diagonal, Integral harmlos ($r=0$ bis 1)

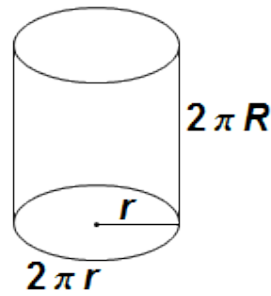
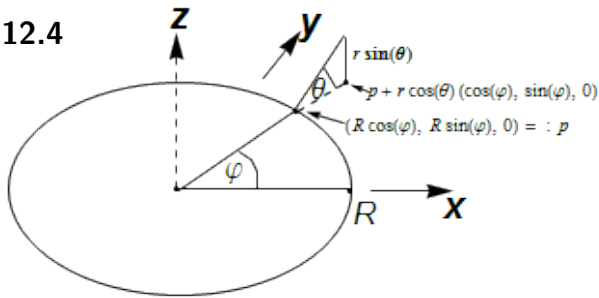
$$g_{ij} = \begin{pmatrix} 1 + 4r^2 & 0 \\ 0 & r^2 \end{pmatrix}, \text{Oberfl.} = \int_0^1 \int_0^{2\pi} \sqrt{1 + 4r^2} d\varphi dr = 2\pi \cdot \frac{2}{3} [(1 + 4r^2)^{\frac{3}{2}}]_0^1 \cdot \frac{1}{8} = \frac{\pi}{6} [5^{\frac{3}{2}} - 1]$$

Alternative: $(x, y) \mapsto (x, y, \underbrace{x^2 + y^2}_{h(x)})$, $\nabla h(x, y) = (2x, 2y)$

Vorlesung: $\sqrt{1 + \|\nabla h\|^2} dxdy = \sqrt{1 + 4x^2 + 4y^2}$

$$\text{Fl.} = \int_{x^2+y^2 \leq 1} \sqrt{\dots} dxdy$$

12.4



$$\text{a) } \partial_1 \psi(\varphi, \theta) = \begin{pmatrix} -R \sin(\varphi) \\ R \cos(\varphi) \\ 0 \end{pmatrix} + \begin{pmatrix} -r \sin(\varphi) \cos(\theta) \\ r \cos(\varphi) \cos(\theta) \\ 0 \end{pmatrix} = (R + r \cos(\theta)) \cdot \begin{pmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \end{pmatrix}$$

$$\partial_2(\varphi, \theta) = r \begin{pmatrix} -\cos(\varphi) \sin(\theta) \\ -\sin(\varphi) \sin(\theta) \\ \cos(\theta) \end{pmatrix}$$

$$g_{ij}^\psi(\varphi, \theta) = \langle \partial_i \psi, \partial_j \psi \rangle(\varphi, \theta) = \begin{pmatrix} (R + r \cos(\theta))^2 & 0 \\ 0 & r^2 \end{pmatrix}$$

$$\text{Oberfl.} = r \int_0^{2\pi} \int_0^{2\pi} (R + r \cos(\theta)) d\theta d\varphi = Rr \cdot 4\pi^2 + r^2 \cdot 2\pi \underbrace{\int_0^{2\pi} \cos(\theta) d\theta}_{=0}$$

12.5

$$\begin{aligned} \text{b) } \vec{v} &= \nabla U, \quad \int_{\gamma} \vec{v} d\vec{s} = \int_a^b \langle \underbrace{\vec{v}(\gamma(t))}_{=\nabla U(\gamma(t))}, \dot{\gamma}(t) \rangle dt = \int_0^b \left\{ \frac{d}{ds} \Big|_{s=t} [s \mapsto U(\gamma(s))] \right\} dt \\ &= U(\gamma(b)) - U(\gamma(a)) \quad (\text{HDI}) \end{aligned}$$