Übungen zur Höheren Quantenmechanik

- Blatt 6

Aufgabe 6

5/5

a. inhomogene Maxwell-Gleichungen:

$$\vec{\nabla}\vec{E} = \frac{\rho}{\epsilon_0} \quad , \quad \vec{\nabla}\times\vec{B} = \mu_0\left(\vec{j} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right) \underset{P6}{\Longleftrightarrow} \partial_\mu F^{\mu\nu} = \mu_0 j^\nu$$

Herleitung der Kontinuitätsgleichung $\vec{\nabla}\vec{j} + \frac{\partial \rho}{\partial t} = 0$

Linke Seite: Betrachte $\partial_\nu \partial_\mu F^{\mu\nu}$ (Setze $\mu_0 = 1 \Rightarrow \epsilon_0 = 1, c = 1$)

$$\begin{aligned}
\partial_\nu \partial_\mu F^{\mu\nu} &= \partial_\nu j^\nu = (\partial_t, \partial_x, \partial_y, \partial_z)^T (j^0, j^x, j^y, j^z)^T \\
&= \partial_t j^0 + \partial_x j^x + \partial_y j^y + \partial_z j^z \\
&= \partial_t \rho + \vec{\nabla}\vec{j}
\end{aligned}$$

(✓)

Rechte Seite: Betrachte $\partial_\nu \underbrace{\begin{pmatrix} \vec{\nabla}\vec{E} \\ \vec{\nabla}\times\vec{B} - \partial_t \vec{E} \end{pmatrix}}_{\text{aus } P6}$

$$\begin{aligned}
\partial_\nu \begin{pmatrix} \vec{\nabla}\vec{E} \\ \vec{\nabla}\times\vec{B} - \partial_t \vec{E} \end{pmatrix} &= (\partial_t, \vec{\nabla})^T (\vec{\nabla}\vec{E}, \vec{\nabla}\times\vec{B} - \partial_t \vec{E})^T \\
&= (\partial_t, \vec{\nabla})^T (\rho, \vec{\nabla}\times\vec{B} - \partial_t \vec{E})^T \\
&= \partial_t \rho + \vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) - \partial_t \vec{\nabla}\vec{E} \\
&= \partial_t \rho + 0 - \partial_t \rho \\
&= 0
\end{aligned}$$

(✓)

b. $\frac{1}{4} F^{\mu\nu} F_{\mu\nu} = \frac{1}{4} \sum_{\mu\nu} F^{\mu\nu} F_{\mu\nu}$

$$\nu = 0: \quad \frac{1}{4} \sum_{\mu 0} F^{\mu 0} F_{\mu 0} = \frac{1}{4} \left(0, -\frac{1}{c}E_x, -\frac{1}{c}E_y, -\frac{1}{c}E_z\right)^T \cdot \left(0, \frac{1}{c}E_x, \frac{1}{c}E_y, \frac{1}{c}E_z\right)^T$$
$$= \frac{1}{4}\left(0 - \frac{1}{c^2}E_x^2 - \frac{1}{c^2}E_y^2 - \frac{1}{c^2}E_z^2\right)$$
$$= \frac{1}{4}\left(-\frac{\vec{E}^2}{c^2}\right)$$

$$\nu = 1: \quad \frac{1}{4} \sum_{\mu 1} F^{\mu 1} F_{\mu 1} = \frac{1}{4}\left(-\frac{1}{c}E_x, 0, -B_z, B_y\right)^T \left(\frac{1}{c}E_x, 0, -B_z, B_y\right)^T$$
$$= \frac{1}{4}\left(-\frac{1}{c^2}E_x^2 + 0 + B_z^2 + B_y^2\right)$$

$$\nu = 2: \quad \frac{1}{4} \sum_{\mu 2} F^{\mu 2} F_{\mu 2} = \frac{1}{4}\left(\frac{1}{c}E_y, B_z, 0, -B_x\right)^T \left(-\frac{1}{c}E_y, B_z, 0, -B_x\right)^T$$
$$= \frac{1}{4}\left(-\frac{1}{c^2}E_y^2 + B_z^2 + B_x^2\right)$$

$$\nu = 3: \quad \frac{1}{4} \sum_{\mu 3} F^{\mu 3} F_{\mu 3} = \frac{1}{4}\left(\frac{1}{c}E_z, -B_y, B_x, 0\right)^T \left(-\frac{1}{c}E_z, -B_y, B_x, 0\right)^T$$
$$= \frac{1}{4}\left(-\frac{1}{c^2}E_z^2 + B_y^2 + B_x^2\right)$$

$$\Longrightarrow \frac{1}{4} F^{\mu\nu} F_{\mu\nu} = \frac{1}{4}\Big(\underbrace{-\frac{1}{c^2}\vec{E}^2}_{\text{von } \nu=0} \underbrace{-\frac{1}{c^2}\vec{E}^2}_{\text{von } \nu=1,2,3} + \underbrace{2\vec{B}^2}_{\text{von } \nu=1,2,3}\Big)$$
$$= \frac{1}{2}\left(\vec{B}^2 - \vec{E}^2\right)$$