2 Übungsblatt von Analysis 2 zum Mittwoch, den 3.11.2010

2.1

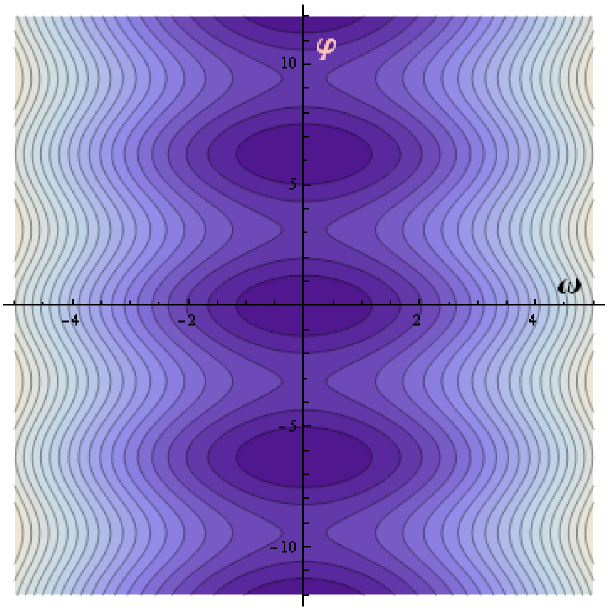
Gauß:

$\int_{\partial Q} < v\nu > dS = \int_Q div(v)dV = \int\limits_0^\pi \int\limits_0^1 \int\limits_0^\pi div(v)dxdydz$
$div(v) = \frac{\partial}{\partial x}(x+z^2) + \frac{\partial}{\partial y}(y\sin(z)) + \frac{\partial}{\partial z}(z^2 + z\cos^2(x)) = 1 + \sin(z) + 2z + \cos^2(x)$
$\int\limits_0^\pi \int\limits_0^1 \int\limits_0^\pi 1 + \sin(z) + 2z + \cos^2(x)dxdydz$
$= \int\limits_0^\pi \int\limits_0^1 \pi\cos^2(x) + \pi^2 + \pi + 2dxdy$
$= \int\limits_0^\pi \pi\cos^2(x) + \pi^2 + \pi + 2dx$
$= 2\pi + \frac{3\pi^2}{2} + \pi^3$

ohne Gauß:

$\phi_1(x,y) = (x,y,0), \phi_{1x} = (1,0,0), \phi_{1y} = (0,1,0),$
$dS_1 = \phi_{1x} \times \phi_{1y}dxdy = (0,0,1)dxdy$ (unten)
$\phi_2(x,y) = (x,y,\pi), \phi_{2x} = (1,0,0), \phi_{2y} = (0,1,0),$
$dS_2 = \phi_{2x} \times \phi_{2y}dxdy = (0,0,1)dxdy$ (oben)
$\phi_3(x,z) = (x,0,z), \phi_{3x} = (1,0,0), \phi_{3z} = (0,0,1),$
$dS_3 = \phi_{3x} \times \phi_{3z}dxdz = (0,1,0)dxdz$ (links)
$\phi_4(x,z) = (x,1,z), \phi_{4x} = (1,0,0), \phi_{4z} = (0,0,1),$
$dS_4 = \phi_{4x} \times \phi_{4z}dxdz = (0,1,0)dxdz$ (rechts)
$\phi_5(y,z) = (0,y,z), \phi_{5y} = (0,1,0), \phi_{5z} = (0,0,1),$
$dS_5 = \phi_{5y} \times \phi_{5z}dydz = (1,0,0)dydz$ (hinten)
$\phi_6(y,z) = (\pi,y,z), \phi_{6y} = (0,1,0), \phi_{6z} = (0,0,1),$
$dS_6 = \phi_{6y} \times \phi_{6z}dydz = (1,0,0)dydz$ (vorne)
$-n_1 = n_2 = (0,0,1),\ -n_3 = n_4 = (0,1,0),\ -n_5 = n_6 = (1,0,0)$
-Vorzeichen für unten, links und hinten, damit entsprechende Normalvektoren von Würfel weg zeigen.
$\int_{\partial Q} < v,\nu > dS = \sum\limits_{i=1}^{6} \int_{\partial Q_i} < v(\phi), \nu_i > dS_i$
$< v(\phi_1), n_1 >= 0^2 + 0\cos^2(x) = 0$
$< v(\phi_2), n_2 >= \pi^2 + \pi\cos^2(x)$
$< v(\phi_3), n_3 >= 0\sin(z)$
$< v(\phi_4), n_4 >= \sin(z)$
$< v(\phi_5), n_5 >= 0 + z^2$
$< v(\phi_6), n_6 >= \pi + z^2$
$\Rightarrow \int_{\partial Q} < v,\nu > dS = \int\limits_0^\pi \int\limits_0^1 (\pi^2 + \pi\cos^2(x))dxdy - \int\limits_0^\pi \int\limits_0^1 (0)dxdy + \int\limits_0^\pi \int\limits_0^\pi (\sin(z))dxdz$
$- \int\limits_0^\pi \int\limits_0^\pi (0)dxdz + \int\limits_0^1 \int\limits_0^\pi (\pi + z^2)dydz - \int\limits_0^1 \int\limits_0^\pi (z^2)dydz$
$= \frac{\pi^2}{2} + \pi^3 - 0 + 2\pi - 0 + \pi^2 + \frac{\pi^3}{3} - \frac{\pi^3}{3} = 2\pi + \frac{3\pi^2}{2} + \pi^3$

2.2



Energieniveaus erfüllen die Gleichung $c = \frac{\omega^2}{2} + \kappa(1 - \cos(\varphi))$.
$\Rightarrow \omega(\varphi) = \pm\sqrt{2(c - \kappa + \kappa\cos(\varphi))}$ Mit $c = 2\kappa \Rightarrow \omega(\varphi) = \pm\sqrt{2(\kappa + \kappa\cos(\varphi))}$ Betrachtet: oberer Teil $(+)$ (x-Achsensymmetrisch, also Winkel unten gleich)
Schnittpunkt mit der φ-Achse bei π, da w=0 wenn $\cos(\phi) = -1$

Taylor-Entwicklung von $1 + \cos(x)$ an $x_0 = \pi$:
$T(x) = 1 + \cos(\pi) - \sin(\pi)(x - \pi) - \frac{1}{2}\cos(\pi)(x - \pi)^2 = \frac{1}{2}(x - \pi)^2$

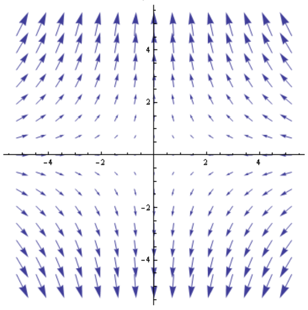
$\Rightarrow \omega_+(\varphi) = \sqrt{2\kappa(\frac{1}{2}(\varphi - \pi)^2)} = \sqrt{\kappa}(\varphi - \pi)$
$\omega'_+(\varphi_0) = \lim\limits_{\varepsilon\to 0} \frac{\omega_+(\varphi_0) - \omega_+(\varphi_0 - \varepsilon)}{\varepsilon} = \lim\limits_{\varepsilon\to 0} \frac{\sqrt{\kappa}(\pi - \pi) - \sqrt{\kappa}(\pi - \varepsilon - \pi)}{\varepsilon} = \frac{\varepsilon\sqrt{\kappa}}{\varepsilon} = \sqrt{\kappa}$

$\Rightarrow \alpha = \arctan(\sqrt{\kappa})$

2.3

a) $\lambda_1 = -1, \quad \lambda_2 = 2$

$$\Rightarrow \xi(t) = \begin{cases} x_0 e^{-t} \\ y_0 e^{2t} \end{cases}$$



b)

2.4

a) (lineare Gl. mit konst. Koeffizienten, inhomogen)
   Lösung: $x(t) = 0e^{\lambda t} + \int\limits_0^t e^{\lambda(t-s)} \sin(s) ds = \frac{e^{t\lambda} - \lambda \sin(t) - \cos(t)}{\lambda^2+1}$

b) (linares Gl. mit var. Koeff., homogen)
   Lösung: $x(t) = exp(\int\limits_0^t \sin(s) ds) \cdot 1 = e^{1-\cos(t)}$

2.5

Zu zeigen: $\forall \varphi \in C_c^\infty([a,b], \mathbb{R}) : \int\limits_a^b f\varphi dx = 0 \Rightarrow f = 0.$
Angenommen$f \neq 0$. O.E. $\exists x_0 \in [a,b] : f(x_0) > 0$
Dann $\exists \varepsilon > 0 : \overline{B_\varepsilon(x_0)} \subset [a,b]$ und $f(x) \geq \frac{1}{2} f(x_0) > 0$ für $||x - x_0|| \leq \varepsilon$
Damit $0 = \int\limits_a^b f(x)\varphi(x) dx = \int\limits_{B_\varepsilon(x_0)} f(x)\varphi(x) dx \geq \frac{1}{2} f(x_0) \int\limits_{B_\varepsilon(x_0)} \varphi(x) dx = \frac{1}{2} f(x_0) > 0$
Also Widerspruch