

2 Übungsblatt von Analysis 2 zum Mittwoch, den 3.11.2010

2.1

Gauß:

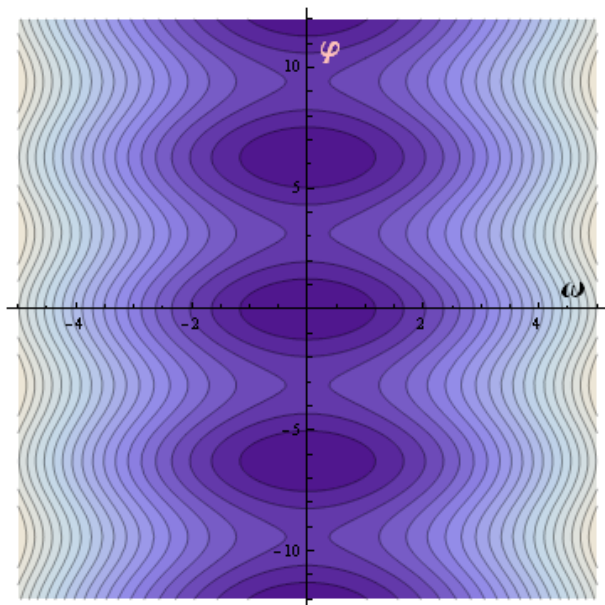
$$\begin{aligned}
 \int_{\partial Q} \langle v, \nu \rangle dS &= \int_Q \operatorname{div}(v) dV = \int_0^\pi \int_0^1 \int_0^\pi \operatorname{div}(v) dx dy dz \\
 \operatorname{div}(v) &= \frac{\partial}{\partial x}(x + z^2) + \frac{\partial}{\partial y}(y \sin(z)) + \frac{\partial}{\partial z}(z^2 + z \cos^2(x)) = 1 + \sin(z) + 2z + \cos^2(x) \\
 \int_0^\pi \int_0^1 \int_0^\pi 1 + \sin(z) + 2z + \cos^2(x) dx dy dz &= \int_0^\pi \int_0^1 \pi \cos^2(x) + \pi^2 + \pi + 2x dy \\
 &= \int_0^\pi \pi \cos^2(x) + \pi^2 + \pi + 2x dx \\
 &= 2\pi + \frac{3\pi^2}{2} + \pi^3
 \end{aligned}$$

ohne Gauß:

$$\begin{aligned}
 \phi_1(x, y) &= (x, y, 0), \phi_{1x} = (1, 0, 0), \phi_{1y} = (0, 1, 0), \\
 dS_1 &= \phi_{1x} \times \phi_{1y} dx dy = (0, 0, 1) dx dy \text{ (unten)} \\
 \phi_2(x, y) &= (x, y, \pi), \phi_{2x} = (1, 0, 0), \phi_{2y} = (0, 1, 0), \\
 dS_2 &= \phi_{2x} \times \phi_{2y} dx dy = (0, 0, 1) dx dy \text{ (oben)} \\
 \phi_3(x, z) &= (x, 0, z), \phi_{3x} = (1, 0, 0), \phi_{3z} = (0, 0, 1), \\
 dS_3 &= \phi_{3x} \times \phi_{3z} dx dz = (0, 1, 0) dx dz \text{ (links)} \\
 \phi_4(x, z) &= (x, 1, z), \phi_{4x} = (1, 0, 0), \phi_{4z} = (0, 0, 1), \\
 dS_4 &= \phi_{4x} \times \phi_{4z} dx dz = (0, 1, 0) dx dz \text{ (rechts)} \\
 \phi_5(y, z) &= (0, y, z), \phi_{5y} = (0, 1, 0), \phi_{5z} = (0, 0, 1), \\
 dS_5 &= \phi_{5y} \times \phi_{5z} dy dz = (1, 0, 0) dy dz \text{ (hinten)} \\
 \phi_6(y, z) &= (\pi, y, z), \phi_{6y} = (0, 1, 0), \phi_{6z} = (0, 0, 1), \\
 dS_6 &= \phi_{6y} \times \phi_{6z} dy dz = (1, 0, 0) dy dz \text{ (vorne)} \\
 -n_1 &= n_2 = (0, 0, 1), -n_3 = n_4 = (0, 1, 0), -n_5 = n_6 = (1, 0, 0) \\
 &\text{-Vorzeichen für unten, links und hinten, damit entsprechende Normalvektoren von Würfel weg zeigen.}
 \end{aligned}$$

$$\begin{aligned}
 \int_{\partial Q} \langle v, \nu \rangle dS &= \sum_{i=1}^6 \int_{\partial Q_i} \langle v(\phi), \nu_i \rangle dS_i \\
 \langle v(\phi_1), n_1 \rangle &= 0^2 + 0 \cos^2(x) = 0 \\
 \langle v(\phi_2), n_2 \rangle &= \pi^2 + \pi \cos^2(x) \\
 \langle v(\phi_3), n_3 \rangle &= 0 \sin(z) \\
 \langle v(\phi_4), n_4 \rangle &= \sin(z) \\
 \langle v(\phi_5), n_5 \rangle &= 0 + z^2 \\
 \langle v(\phi_6), n_6 \rangle &= \pi + z^2 \\
 \Rightarrow \int_{\partial Q} \langle v, \nu \rangle dS &= \int_0^\pi \int_0^1 (\pi^2 + \pi \cos^2(x)) dx dy - \int_0^\pi \int_0^1 (0) dx dy + \int_0^\pi \int_0^\pi (\sin(z)) dx dz \\
 &\quad - \int_0^\pi \int_0^\pi (0) dx dz + \int_0^1 \int_0^\pi (\pi + z^2) dy dz - \int_0^1 \int_0^\pi (z^2) dy dz \\
 &= \frac{\pi^2}{2} + \pi^3 - 0 + 2\pi - 0 + \pi^2 + \frac{\pi^3}{3} - \frac{\pi^3}{3} = 2\pi + \frac{3\pi^2}{2} + \pi^3
 \end{aligned}$$

2.2



Energieniveaus erfüllen die Gleichung $c = \frac{\omega^2}{2} + \kappa(1 - \cos(\varphi))$.

$\Rightarrow \omega(\varphi) = \pm \sqrt{2(c - \kappa + \kappa \cos(\varphi))}$ Mit $c = 2\kappa \Rightarrow \omega(\varphi) = \pm \sqrt{2(\kappa + \kappa \cos(\varphi))}$ Betrachtet: oberer Teil (+) (x-Achsensymmetrisch, also Winkel unten gleich)

Schnittpunkt mit der φ -Achse bei π , da $w=0$ wenn $\cos(\phi) = -1$

Taylor-Entwicklung von $1 + \cos(x)$ an $x_0 = \pi$:

$$T(x) = 1 + \cos(\pi) - \sin(\pi)(x - \pi) - \frac{1}{2} \cos(\pi)(x - \pi)^2 = \frac{1}{2}(x - \pi)^2$$

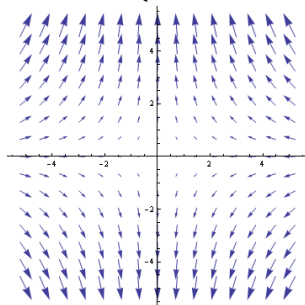
$$\Rightarrow \omega_+(\varphi) = \sqrt{2\kappa(\frac{1}{2}(\varphi - \pi)^2)} = \sqrt{\kappa}(\varphi - \pi)$$

$$\omega'_+(\varphi_0) = \lim_{\varepsilon \rightarrow 0} \frac{\omega_+(\varphi_0) - \omega_+(\varphi_0 - \varepsilon)}{\varepsilon} = \lim_{\varepsilon \rightarrow 0} \frac{\sqrt{\kappa}(\pi - \pi) - \sqrt{\kappa}(\pi - \varepsilon - \pi)}{\varepsilon} = \frac{\varepsilon \sqrt{\kappa}}{\varepsilon} = \sqrt{\kappa}$$

$$\Rightarrow \alpha = \arctan(\sqrt{\kappa})$$

2.3

a) $\lambda_1 = -1, \lambda_2 = 2$
 $\Rightarrow \xi(t) = \begin{cases} x_0 e^{-t} \\ y_0 e^{2t} \end{cases}$



b)

2.4

a) (lineare Gl. mit konst. Koeffizienten, inhomogen)

$$\text{Lösung: } x(t) = 0e^{\lambda t} + \int_0^t e^{\lambda(t-s)} \sin(s) ds = \frac{e^{t\lambda} - \lambda \sin(t) - \cos(t)}{\lambda^2 + 1}$$

b) (lineares Gl. mit var. Koeff., homogen)

$$\text{Lösung: } x(t) = \exp\left(\int_0^t \sin(s) ds\right) \cdot 1 = e^{1 - \cos(t)}$$

2.5

Zu zeigen: $\forall \varphi \in C_c^\infty([a, b], \mathbb{R}) : \int_a^b f \varphi dx = 0 \Rightarrow f = 0$.

Angenommen $f \neq 0$. O.E. $\exists x_0 \in [a, b] : f(x_0) > 0$

Dann $\exists \varepsilon > 0 : \overline{B_\varepsilon(x_0)} \subset [a, b]$ und $f(x) \geq \frac{1}{2}f(x_0) > 0$ für $\|x - x_0\| \leq \varepsilon$

$$\text{Damit } 0 = \int_a^b f(x) \varphi(x) dx = \int_{B_\varepsilon(x_0)} f(x) \varphi(x) dx \geq \frac{1}{2}f(x_0) \int_{B_\varepsilon(x_0)} \varphi(x) dx = \frac{1}{2}f(x_0) > 0$$

Also Widerspruch