Numerik 1   Blatt 6   Julian Bergmann

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Aufgabe 1

Lemma v. Gershgorin:

Skript

$$\forall \lambda_j \text{ EW v. } A: \quad \lambda_j \in K_j := \{ z \in \mathbb{C} \mid |z - a_{jj}| \le \sum_{\substack{k=1\\k\ne j}}^{n} |a_{jk}| \wedge |z-a_{jj}| \le \sum_{\substack{k=1\\k\ne j}}^{n} |a_{kj}| \}$$

Vorlesung

$$\text{Matrix } A \text{ symmetrisch: } \sum_{\substack{k=1\\k\ne j}}^{n} |a_{jk}| = \sum_{\substack{k=1\\k\ne j}}^{n} |a_{kj}|$$

$$|\lambda_a - a_{11}| = |\lambda_a - 7| \le |\tfrac{1}{4}| + |\tfrac{1}{2}| = \tfrac{3}{4} \quad \Leftrightarrow \lambda_a \in [7-\tfrac{3}{4}, 7+\tfrac{3}{4}]$$

$$|\lambda_b - a_{22}| = |\lambda_b + 3| \le |1| + |\tfrac{1}{4}| = \tfrac{5}{4} \quad \Leftrightarrow \lambda_b \in [-3-\tfrac{5}{4}, -3+\tfrac{5}{4}]$$

$$|\lambda_c - a_{33}| = |\lambda_c - \tfrac{3}{2}| \le |1| + |\tfrac{1}{2}| = \tfrac{3}{2} \quad \Leftrightarrow \lambda_c \in [\tfrac{3}{2}-\tfrac{3}{2}, \tfrac{3}{2}+\tfrac{3}{2}]$$ ✓

$$\lambda_1 < \lambda_2 < \lambda_3 \quad \Leftrightarrow \quad \lambda_1 = \lambda_b, \quad \lambda_2 = \lambda_c, \quad \lambda_3 = \lambda_a$$

kl. EW: $\lambda_1 \in [-\frac{17}{4}, -\frac{7}{4}]$

Inverse Iteration:

$$(A - \lambda \mathbb{1})\, x^{(k+1)} = x^{(k)} \qquad , \lambda \text{ als Näherung } = -3$$

$$[A + 3\cdot\mathbb{1} \mid x^{(0)}] = \left(\begin{array}{ccc|c} 10 & 1/4 & 1/2 & -1 \\ 1/4 & 0 & 1 & 0 \\ 1/2 & 1 & 9/2 & 0 \end{array}\right)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 1/40 & 1 & 0 \\ 1/20 & 0 & 1 \end{pmatrix} \left(\begin{array}{ccc|c} 10 & 1/4 & 1/2 & -1 \\ 0 & -1/160 & \frac{79}{80} & 1/40 \\ 0 & \frac{79}{80} & \frac{179}{40} & 1/20 \end{array}\right) = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1/40 & 1 & 0 \\ 1/20 & -158 & 1 \end{pmatrix}}_{L} \left(\begin{array}{ccc|c} 10 & 1/4 & 1/2 & -1 \\ 0 & -1/160 & \frac{72}{80} & 1/40 \\ 0 & 0 & \frac{321}{2} & 4 \end{array}\right)$$

$$ \underbrace{}_{R} \quad \underbrace{}_{\tilde{x}^{(1)}}$$

$$R\,\tilde{\tilde{x}}^{(1)} = \tilde{x}^{(1)}: \quad \left(\begin{array}{ccc|c} 10 & 1/4 & 1/2 & -1 \\ 0 & -1/160 & \frac{79}{80} & 1/40 \\ 0 & 0 & \frac{321}{2} & 4 \end{array}\right) \Leftrightarrow \left(\begin{array}{ccc|c} 10 & 1/4 & 0 & -\frac{325}{321} \\ 0 & -1/160 & 0 & 1/2568 \\ 0 & 0 & 1 & 8/321 \end{array}\right)$$ ✓

$$\Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & -32/321 \\ 0 & 1 & 0 & -20/321 \\ 0 & 0 & 1 & 8/321 \end{array}\right)$$

$$x^{(1)} = \frac{\tilde{\tilde{x}}^{(1)}}{\|\tilde{\tilde{x}}^{(1)}\|_2} = \left(-\frac{32}{321}, -\frac{20}{321}, \frac{8}{321}\right)^T \cdot \frac{1}{0{,}1202} =$$

$$= (-0{,}8296, -0{,}5185, 0{,}2074)^T$$

$$L\,\tilde{x}^{(2)} = x^{(1)}: \quad \left(\begin{array}{ccc|c} 1 & 0 & 0 & -0{,}8296 \\ 1/40 & 1 & 0 & -0{,}5185 \\ 1/20 & -158 & 1 & 0{,}2074 \end{array}\right) \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & -0{,}8296 \\ 0 & 1 & 0 & -0{,}4977 \\ 0 & 0 & 1 & \underbrace{-78{,}388}_{\tilde{x}^{(2)}} \end{array}\right)$$

$$R\,\tilde{\tilde{x}}^{(2)} = \tilde{x}^{(2)}: \quad \left(\begin{array}{ccc|c} 10 & 1/4 & 1/2 & -0{,}8296 \\ 0 & -1/160 & \frac{79}{80} & -0{,}4977 \\ 0 & 0 & \frac{321}{2} & -78{,}388 \end{array}\right) \Leftrightarrow \left(\begin{array}{ccc|c} 10 & 1/4 & 0 & -0{,}5854 \\ 0 & -1/160 & 0 & -0{,}0154 \\ 0 & 0 & 1 & -0{,}4884 \end{array}\right)$$

$$\Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & -0{,}1201 \\ 0 & 1 & 0 & 2{,}464 \\ 0 & 0 & 1 & -0{,}4884 \end{array}\right)$$

$$x^{(2)} = \frac{\tilde{\tilde{x}}^{(2)}}{\|\tilde{\tilde{x}}^{(2)}\|_2} = (-0{,}1201\ ,\ 2{,}464\ ,\ -0{,}4884)^T \cdot \frac{1}{2{,}5148}$$

$$= (-0{,}04776\ ,\ 0{,}9798\ ,\ -0{,}19421)^T$$

$$L\,\tilde{x}^{(3)} = x^{(2)} : \quad \left(\begin{array}{ccc|c} 1 & 0 & 0 & -0{,}04776 \\ \frac{1}{40} & 1 & 0 & 0{,}9798 \\ \frac{1}{20} & -158 & 1 & -0{,}19421 \end{array}\right) \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & -0{,}04776 \\ 0 & 1 & 0 & 0{,}98099 \\ 0 & 0 & 1 & 154{,}805 \end{array}\right)$$

$$R\,\tilde{\tilde{x}}^{(3)} = \tilde{x}^{(3)} : \quad \left(\begin{array}{ccc|c} 10 & \frac{1}{4} & \frac{1}{2} & -0{,}04776 \\ 0 & -\frac{1}{160} & \frac{79}{80} & 0{,}58099 \\ 0 & 0 & \frac{321}{2} & 154{,}805 \end{array}\right) \Leftrightarrow \left(\begin{array}{ccc|c} 10 & \frac{1}{4} & 0 & -0{,}53002 \\ 0 & -\frac{1}{160} & 0 & 0{,}02853 \\ 0 & 0 & 1 & 0{,}96452 \end{array}\right)$$

$$\Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0{,}06112 \\ 0 & 1 & 0 & -4{,}5648 \\ 0 & 0 & 1 & 0{,}96452 \end{array}\right)$$

$$x^{(3)} = \frac{\tilde{\tilde{x}}^{(3)}}{\|x^{(3)}\|_2} = (0{,}06112\ ,\ -4{,}5648\ ,\ 0{,}96452)^T \cdot \frac{1}{4{,}666}$$

$$= (0{,}0131\ ,\ -0{,}97831\ ,\ 0{,}20671)^T$$

$$\lambda = \frac{x^{(3)T} A\, x^{(3)}}{\underbrace{x^{(3)T} x^{(3)}}_{=1 \text{ da Normiert}}} = x^{(3)T} A\, x^{(3)}$$

$$A\,x^{(3)} = \begin{pmatrix} 7 & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & -3 & 1 \\ \frac{1}{2} & 1 & \frac{3}{2} \end{pmatrix} \begin{pmatrix} 0{,}0131 \\ -0{,}97831 \\ 0{,}20671 \end{pmatrix} = \begin{pmatrix} -0{,}04952 \\ 3{,}14491 \\ -0{,}66563 \end{pmatrix}$$

$$x^{(3)T} x^{(3)} = (0{,}0131\ ,\ -0{,}97831\ ,\ 0{,}20671) \begin{pmatrix} -0{,}04952 \\ 3{,}14491 \\ -0{,}66563 \end{pmatrix}$$

$$= -3{,}215 \quad ✓$$

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Physik Mittwoch

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Aufgabe 2

$$A = \begin{pmatrix} -5{,}7 & -61{,}1 & -32{,}9 \\ 0{,}8 & 11{,}9 & 7{,}1 \\ -1{,}1 & -11{,}8 & -7{,}2 \end{pmatrix}$$

EV w_1 zu $\lambda_1 = 4$: $(A - \mathbb{1}\lambda_1) w_1 = \vec{0}$

$$\Rightarrow \begin{pmatrix} -9{,}7 & -61{,}1 & -32{,}9 \\ 0{,}8 & 7{,}9 & 7{,}1 \\ -1{,}1 & -11{,}8 & -11{,}2 \end{pmatrix} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{w_1} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \left(\begin{array}{ccc|c} -9{,}7 & -61{,}1 & -32{,}9 & 0 \\ 0 & 2{,}86 & 4{,}39 & 0 \\ 0 & -4{,}87 & -7{,}47 & 0 \end{array}\right) \Leftrightarrow \left(\begin{array}{ccc|c} -9{,}7 & -61{,}1 & -32{,}9 & 0 \\ 0 & 2{,}86 & 4{,}39 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Leftrightarrow \left| \begin{array}{r} -9{,}7x - 61{,}1y - 32{,}9z = 0 \\ 2{,}86y + 4{,}39z = 0 \end{array} \right| \Leftrightarrow \left| \begin{array}{l} x = 6{,}277\,z \\ y = -1{,}535\,z \end{array} \right|$$

$$\Rightarrow w_1 = (6{,}277\,, -1{,}535\,, 1)^T \underset{\text{normiert}}{\Longrightarrow} w_1 = (0{,}96\,, -0{,}235\,, 0{,}153)^T$$

$\|w_1\|_2 = 6{,}539$

✓

De flation :

$$SAS^{-1} = \begin{pmatrix} \lambda_1 & * \cdots * \\ 0 & \\ \vdots & B \\ 0 & \end{pmatrix}$$

S über Householder :

$$u = \frac{w_1 - \alpha e_1}{\|w_1 - \alpha e_1\|_2} \quad , \quad \alpha = \pm \|w_1\|_2 \quad , \quad w_{11} > 0 \Rightarrow \alpha = -\|w_1\|_2$$

$$\Rightarrow u = \frac{w_1 + \|w_1\|_2 e_1}{\|w_1 + \|w_1\|_2 e_1\|_2} = \begin{pmatrix} 0{,}96 + 1 \\ -0{,}235 \\ 0{,}153 \end{pmatrix} \cdot \frac{1}{1{,}98} = \begin{pmatrix} 0{,}99 \\ -0{,}119 \\ 0{,}077 \end{pmatrix}$$

$$S = \mathbb{1} - 2uu^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - 2 \begin{pmatrix} 0{,}98 & -0{,}117 & 0{,}0765 \\ -0{,}117 & 0{,}0141 & -0{,}0092 \\ 0{,}0765 & -0{,}0092 & 0{,}00597 \end{pmatrix}$$

$$= \begin{pmatrix} -0{,}96 & 0{,}235 & -0{,}153 \\ 0{,}235 & 0{,}972 & 0{,}0183 \\ -0{,}153 & 0{,}0183 & 0{,}988 \end{pmatrix} = S = S^{-1}$$

✓

$$AS^{-1} = \begin{pmatrix} -5{,}7 & -61{,}1 & -32{,}9 \\ 0{,}8 & 11{,}9 & 7{,}1 \\ -1{,}1 & -11{,}8 & -7{,}2 \end{pmatrix} \begin{pmatrix} -0{,}96 & 0{,}235 & -0{,}153 \\ 0{,}235 & 0{,}972 & 0{,}0183 \\ -0{,}153 & 0{,}0183 & 0{,}988 \end{pmatrix} = \begin{pmatrix} -3{,}84 & -61{,}32 & -32{,}75 \\ 0{,}94 & 11{,}88 & 7{,}11 \\ -0{,}673 & -11{,}86 & -7{,}16 \end{pmatrix}$$

$$SAS^{-1} = \begin{pmatrix} -0{,}96 & 0{,}235 & -0{,}153 \\ 0{,}235 & 0{,}972 & 0{,}0183 \\ -0{,}153 & 0{,}0183 & 0{,}988 \end{pmatrix} \begin{pmatrix} -3{,}84 & -61{,}32 & -32{,}75 \\ 0{,}94 & 11{,}88 & 7{,}11 \\ -0{,}673 & -11{,}86 & -7{,}16 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 63{,}47 & 34{,}2 \\ 0 & -3{,}06 & -0{,}91 \\ 0 & -2{,}12 & -1{,}94 \end{pmatrix}$$

✓

$\Rightarrow$

Kleinster EW:

Abschätzung über Gersgorin:

$$B = \begin{pmatrix} -3{,}06 & -0{,}91 \\ -2{,}12 & -1{,}94 \end{pmatrix} \Rightarrow \begin{matrix} |\lambda + 3{,}06| \leq 0{,}91 \\ |\lambda + 1{,}94| \leq 0{,}91 \end{matrix}$$

$\Rightarrow$ 1. Nährung kl. $|\lambda_{min}| \approx 1{,}94$, Betragsmäßig kl. EW auch über $\lambda_{min} \approx 0$!

Inverse Iteration:

$$B' = B - \mathbb{1} \cdot 0 = \begin{pmatrix} -3{,}06 & -0{,}91 \\ -2{,}12 & -1{,}94 \end{pmatrix} \qquad x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$B' = \begin{pmatrix} 1 & 0 \\ 0{,}693 & 1 \end{pmatrix} \begin{pmatrix} -3{,}06 & -0{,}91 \\ 0 & -1{,}31 \end{pmatrix}$$

$$L\hat{x}_1 = x_0 : \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0{,}693 & 1 & 0 \end{array}\right) \Leftrightarrow \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -0{,}693 \end{array}\right) \quad ✓$$

$$R\tilde{\hat{x}}_1 = \hat{x}_1 : \left(\begin{array}{cc|c} -3{,}06 & -0{,}91 & 1 \\ 0 & -1{,}31 & -0{,}693 \end{array}\right) \Leftrightarrow \left(\begin{array}{cc|c} 1 & 0 & -0{,}484 \\ 0 & 1 & 0{,}529 \end{array}\right) \quad ✓$$

$$x_1 = \frac{\tilde{\hat{x}}_1}{\|\tilde{\hat{x}}_1\|_2} = \begin{pmatrix} -0{,}484 \\ 0{,}529 \end{pmatrix} \cdot \frac{1}{0{,}717} = \begin{pmatrix} -0{,}675 \\ 0{,}738 \end{pmatrix} \quad ✓$$

$$L\hat{x}_2 = x_1 : \left(\begin{array}{cc|c} 1 & 0 & -0{,}675 \\ 0{,}693 & 1 & 0{,}738 \end{array}\right) \Leftrightarrow \left(\begin{array}{cc|c} 1 & 0 & -0{,}675 \\ 0 & 1 & 1{,}205 \end{array}\right)$$

$$R\tilde{\hat{x}}_2 = \hat{x}_2 : \left(\begin{array}{cc|c} -3{,}06 & -0{,}91 & -0{,}675 \\ 0 & -1{,}31 & 1{,}205 \end{array}\right) \Leftrightarrow \left(\begin{array}{cc|c} 1 & 0 & 0{,}494 \\ 0 & 1 & -0{,}921 \end{array}\right) \quad ✓$$

$$x_2 = \frac{\tilde{\hat{x}}_2}{\|\tilde{\hat{x}}_2\|_2} = \begin{pmatrix} 0{,}494 \\ -0{,}921 \end{pmatrix} \cdot \frac{1}{1{,}045} = \begin{pmatrix} 0{,}473 \\ -0{,}881 \end{pmatrix}$$

$$\lambda = x_2^T B x_2 = \begin{pmatrix} 0{,}473 \\ -0{,}881 \end{pmatrix}^T \begin{pmatrix} -3{,}06 & -0{,}91 \\ -2{,}12 & -1{,}94 \end{pmatrix} \begin{pmatrix} 0{,}473 \\ -0{,}881 \end{pmatrix}$$

$$= (0{,}473 \,,\, -0{,}881) \begin{pmatrix} -0{,}646 \\ 0{,}706 \end{pmatrix} = -0{,}928 \quad (✓)$$

$$\lambda = \frac{1}{-0{,}928} \quad (-1)$$

$\|x_1 - x_0\|_2 = 1{,}83$

$\|x_2 - x_1\|_2 = 1{,}98$

$\Rightarrow$ λ nur grob richtig, da Abstände nur grob konvergieren.