Numerik 1 Blatt 7 Julian Bergmann
Physik Mittwoch

Aufgabe 21

9,5

a)

x	-2	-1	0	1	2
y	2	1	0	1	2

$$L_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} = \frac{(x+1)(x-0)(x-1)(x-2)}{(-2+1)(-2+0)(-2-1)(-2-2)} = \frac{x^4-2x^3-x^2+2x}{24}$$

$$L_1(x) = \frac{(x+2)(x-0)(x-1)(x-2)}{(-1+2)(-1-0)(-1-1)(-1-2)} = -\frac{x^4-x^3-4x^2+4x}{6}$$

$$L_2(x) = \frac{(x+2)(x+1)(x-1)(x-2)}{(0+2)(0+1)(0-1)(0-2)} = \frac{x^4-5x^2+4}{4}$$

$$L_3(x) = \frac{(x+2)(x+1)(x-0)(x-2)}{(1+2)(1+1)(1-0)(1-2)} = -\frac{x^4+x^3-4x^2-4x}{6}$$

$$L_4(x) = \frac{(x+2)(x+1)(x-0)(x-1)}{(2+2)(2+1)(2-0)(2-1)} = \frac{x^4+2x^3-x^2-2x}{24}$$

$$p_a(x) = \sum_{i=0}^{n} y_i L_i(x) = \frac{x^4-2x^3-x^2+2x}{12} - \frac{x^4-x^3-4x^2+4x}{6} - \frac{x^4+x^3-4x^2-4x}{6} + \frac{x^4+2x^3-x^2-2x}{12}$$

$$= x^4\left(\frac{1}{12}-\frac{1}{6}-\frac{1}{6}+\frac{1}{12}\right) + x^3\left(-\frac{1}{6}+\frac{1}{6}-\frac{1}{6}+\frac{1}{6}\right) + x^2\left(-\frac{1}{12}+\frac{2}{3}+\frac{2}{3}-\frac{1}{12}\right) + x\left(\frac{1}{6}-\frac{2}{3}+\frac{2}{3}-\frac{1}{6}\right)$$

$$= -\frac{1}{6}x^4 + \frac{7}{6}x^2$$

b) $p_b(x) = u_1(x) + u_2(x) + u_3(x) = c_1 + c_2 x + c_3 x^2$

x	-2	-1,5	-1	-0,5	0	0,5	1	1,5	2	
y	2	1,5	1	0,5	0	0,5	1	1,5	2	$=: b$

$$A = \begin{pmatrix} 1 & -2 & 4 \\ 1 & -1{,}5 & 2{,}25 \\ 1 & -1 & 1 \\ 1 & -0{,}5 & 0{,}25 \\ 1 & 0 & 0 \\ 1 & 0{,}5 & 0{,}25 \\ 1 & 1 & 1 \\ 1 & 1{,}5 & 2{,}25 \\ 1 & 2 & 4 \end{pmatrix} \qquad A^TA = \begin{pmatrix} 9 & 0 & 15 \\ 0 & 15 & 0 \\ 15 & 0 & 44{,}25 \end{pmatrix} \qquad A^Tb = \begin{pmatrix} 10 \\ 0 \\ 25 \end{pmatrix}$$

$$A^TAc = A^Tb \quad (\Leftrightarrow) \left(\begin{array}{ccc|c} 9 & 0 & 15 & 10 \\ 0 & 15 & 0 & 0 \\ 15 & 0 & 44{,}25 & 25 \end{array}\right) (\Leftrightarrow) \left(\begin{array}{ccc|c} 9 & 0 & 15 & 10 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 19{,}25 & \frac{25}{3} \end{array}\right)$$

$$(\Leftrightarrow) \left(\begin{array}{ccc|c} 1 & 0 & 0 & 30/77 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 100/231 \end{array}\right)$$

$$\Rightarrow p_b(x) = \frac{100}{231}x^2 + \frac{30}{77}$$

$\Rightarrow$

$P_a(0) - |0| = 0$ ✓

$P_a(\frac{1}{3}) - |\frac{1}{3}| = -0{,}206$ (✓)

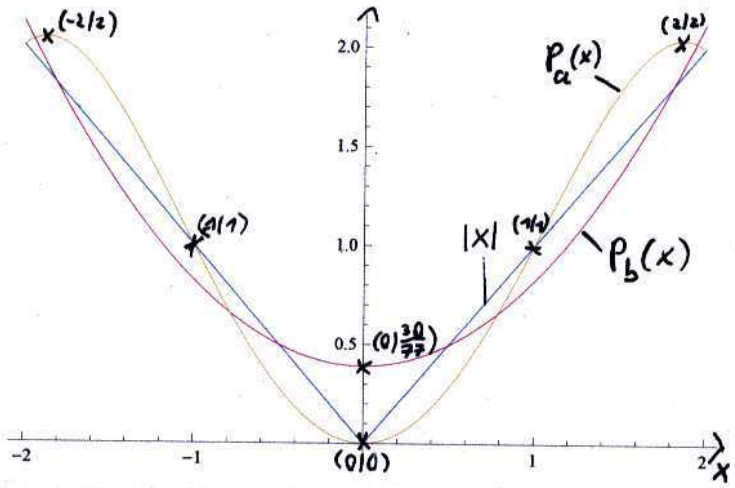
$P_a(\frac{3}{2}) - |\frac{3}{2}| = 0{,}281$ ✓

$P_b(0) - |0| = 0{,}39$ ✓

$P_b(\frac{1}{3}) - |\frac{1}{3}| = 0{,}104$ ✓

$|P_b(\frac{3}{2}) - |\frac{3}{2}|| = |-0{,}136|$ (✓) $-\frac{1}{2}$

⇒ Nahe 0 ist $P_a(x)$ genauer, ab $\frac{1}{3}$ bis $\frac{3}{2}$ $P_b(x)$. Oberflächlich betrachtet sind beide Funktionen im Bereich -2 bis 2 ähnlich genau, P_b erfordert jedoch die doppelte Anzahl an Stützstellen.



Aufgabe 22

(20)

a)

x	0	1	2	4
y	1	3	3	4

$$L_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = \frac{(x-1)(x-2)(x-4)}{(0-1)(0-2)(0-4)} = \frac{x^3-7x^2+14x-8}{-8}$$

$$L_1(x) = \frac{(x-0)(x-2)(x-4)}{(1-0)(1-2)(1-4)} = \frac{x^3-6x^2+8x}{3}$$

$$L_2(x) = \frac{(x-0)(x-1)(x-4)}{(2-0)(2-1)(2-4)} = \frac{x^3-5x^2+4x}{-4}$$

$$L_3(x) = \frac{(x-0)(x-1)(x-2)}{(4-0)(4-1)(4-2)} = \frac{x^3-3x^2+2x}{24}$$

$$P_a(x) = \sum_{i=0}^{3} y_i L_i(x) = -\frac{x^3-7x^2+14x-8}{8} + 3\cdot\frac{x^3-6x^2+8x}{3} - 3\cdot\frac{x^3-5x^2+4x}{4} + 4\cdot\frac{x^3-3x^2+2x}{24}$$

$$= x^3\left(-\frac{1}{8}+1-\frac{3}{4}+\frac{1}{6}\right) + x^2\left(\frac{7}{8}-6+\frac{15}{4}-\frac{1}{2}\right) + x\left(-\frac{14}{8}+8-3+\frac{1}{3}\right) + 1$$

$$= \frac{7}{24}x^3 - \frac{15}{8}x^2 + \frac{43}{12}x + 1$$

$$P_a(5) = \frac{17}{2} = 8{,}5$$

b)

x	0	1	2	4	6
y	1	3	3	4	5

$$L_0(x) = \frac{x^3-7x^2+14x-8}{-8}\cdot\frac{(x-6)}{(0-6)} = \frac{x^4-13x^3+56x^2-92x+48}{48}$$

$$L_1(x) = \frac{x^3-6x^2+8x}{3}\cdot\frac{(x-6)}{(1-6)} = \frac{x^4-12x^3+44x^2-48x}{-15}$$

$$L_2(x) = \frac{x^3-5x^2+4x}{-4}\cdot\frac{(x-6)}{(2-6)} = \frac{x^4-11x^3+34x^2-24x}{16}$$

$$L_3(x) = \frac{x^3-3x^2+2x}{24}\cdot\frac{(x-6)}{(4-6)} = \frac{x^4-9x^3+20x^2-12x}{-48}$$

$$L_4(x) = \frac{(x-0)(x-1)(x-2)(x-4)}{(6-0)(6-1)(6-2)(6-4)} = \frac{x^4-7x^3+14x^2-8x}{240}$$

$$P_b(x) = \sum_{i=0}^{4} y_i L_i(x) = \frac{x^4-13x^3+56x^2-92x+48}{48} - 3\cdot\frac{x^4-12x^3+44x^2-48x}{15} + 3\cdot\frac{x^4-11x^3+34x^2-24x}{16} - 4\cdot\frac{x^4-9x^3+20x^2-12x}{48} + 5\cdot\frac{x^4-7x^3+14x^2-8x}{240}$$

$$= x^4\left(\frac{1}{48}-\frac{1}{5}+\frac{3}{16}-\frac{1}{12}+\frac{1}{48}\right) + x^3\left(-\frac{13}{48}+\frac{12}{5}-\frac{33}{16}+\frac{3}{4}-\frac{7}{48}\right) + x^2\left(\frac{7}{6}-\frac{44}{5}+\frac{51}{8}-\frac{5}{3}+\frac{7}{24}\right) + x\left(-\frac{23}{12}+\frac{48}{5}-\frac{9}{2}+1-\frac{1}{6}\right)+1$$

$$= -\frac{13}{240}x^4 + \frac{161}{240}x^3 - \frac{79}{30}x^2 + \frac{241}{60}x + 1$$

$$\Rightarrow P_b(5) = \frac{21}{4} = 5{,}25$$