Numerik 1 Blatt 10 Julian Bergmann

Aufgabe 1

a) Trapez-Regel: $I(f) = \frac{h}{2}\left(f(a)+f(b)+2\sum_{i=1}^{N-1} f(a+hi)\right)$

$h=\frac{1}{3},\ a=0,\ b=2,\ N=\frac{2-0}{1/3}=6$

$\Rightarrow I\left(\frac{1}{4+x^2}\right) = \left(\frac{1}{4} + \frac{1}{4+2^2} + 2\left(\sum_{i=1}^{5} \frac{1}{(\frac{1}{3}\cdot i)^2+4}\right)\right)\cdot\frac{1}{6}$

$= 0{,}39212$

Simpson-Regel:

$I(f) = \frac{h}{3}\left(\frac{1}{2} f(a) + \sum_{k=1}^{N-1} f(a+kh) + 2\sum_{k=1}^{N} f\left(a+h\left(k-\frac{1}{2}\right)\right) + \frac{1}{2} f(b)\right)$

$\Rightarrow I\left(\frac{1}{4+x^2}\right) = \frac{1}{9}\left(\frac{1}{8}+\frac{1}{16}+\sum_{k=1}^{5}\left(\frac{1}{4+(kh)^2}\right) + 2\sum_{k=1}^{6}\left(\frac{1}{4+(h(k-\frac{1}{2}))^2}\right)\right)$

$= 0{,}392699$

$\pi/8 \approx 0{,}392699$

$\Rightarrow$ Simpson-Regel geringfügig genauer, aber aufwändiger (2 Summen über $\sim N$ statt 1).

b) Fehlerabschätzung (Restglied) zusammengesetzte Trapez-Regel:

$|E(f)| \le \frac{(b-a)}{12} h^2 \max_{a\le x\le b}(|f''(x)|)$

$f(x)=\frac{1}{4+x^2}$, $f'(x)=\frac{-2x}{(4+x^2)^2}$, $f''(x) = \frac{-2(4+x^2)^2+2x\cdot 2(4+x^2)\cdot 2x}{(4+x^2)^4} = \frac{-8+6x^2}{(4+x^2)^3}$

$f'''(x) = \frac{12x(4+x^2)^3 - (-8+6x^2)\cdot 3(4+x^2)^2\cdot 2x}{(4+x^2)^6}$

$= \frac{12x(4+x^2)-(-8+6x^2)\cdot 6}{(4+x^2)^6}$

$= \frac{(-96x+24x^3)(4+x^2)^2}{(4+x^2)^6} = \frac{24x(-4+x^2)}{(4+x^2)^4}$

Maxima:
Minima: $f'''(x_0) \overset{!}{=} 0 = 24x(-4+x^2) \Rightarrow x = \pm 2, 0$

$f''(2) = \frac{-8+6\cdot 4}{(4+4)^3} = \frac{1}{32}$ $\Rightarrow \max_{0\le x\le 2}|f''(x)| = \frac{1}{32}$

$f''(-2) = \frac{-8+6\cdot 4}{(4+4)^3} = \frac{1}{32}$

$f''(0) = \frac{-8}{4^3} = -\frac{1}{8}$ ← Minimum

$\Rightarrow |E(f)| \le \frac{2-0}{12}\cdot\frac{1}{9}\cdot\frac{1}{32} = 0{,}0005787$

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Aufgabe 2

a) $s \in S_{X,k} \Rightarrow x_0 < x_1 < \ldots < x_m$

$* \quad x_{-k+1} = x_{-k+2} = \ldots = x_0 < x_1 < \ldots < x_m = x_{m+1} = \ldots = x_{m+k-1}$

Sei: $A = (B_j(y_i))_{i=1,\ j=-k}^{m+k \quad m-1}$ (Matrix $(m+k)\times(m+k)$)

dabei: $s(y_i) = \sum_{j=-k}^{m-1} \lambda_j B_j(y_i) = b_i$

Das Interpolationsproblem ist genau dann eindeutig Lösbar, wenn $\det(A) \neq 0$!

Nach Schönberg/Whitney:

$\det(A) \neq 0 \Leftrightarrow x_{j-k-1} < y_j < x_j \quad \forall j = 1, \ldots, m+k$

Für $j = 1$:

$x_{j-k-1} = x_{-k} \overset{\text{Vorr.}}{<} x_{-k+1} \overset{*}{=} x_0 \;\mathrm{I}\; y_j = (\underbrace{\overset{x_0}{\widehat{x_{1-k}}} + \ldots + x_0}_{k x_0}) \frac{1}{k} = x_0$

$y_j = x_0 < x_1$

$j = 2$:

$x_{j-k-1} = x_{1-k} \overset{*}{=} x_0 < y_j = (\underbrace{x_{2-k} + \ldots + x_1}_{k x_0 + x_1 - x_0}) \frac{1}{k} = x_0 + \frac{x_1 - x_0}{k}$

$y_j = (\underbrace{x_{2-k}}_{< x_1} + \ldots + x_1) \frac{1}{k} < k x_1 \cdot \frac{1}{k} = x_1 < x_2$

Analog bis $x_{j-k-1} \leq x_0 \Leftrightarrow j = k+1$

$j = k+1$:

$x_{j-k-1} = x_0 < y_j = (x_1 + \ldots + x_k) \frac{1}{k} = k x_1 + \underbrace{\frac{(x_2 - x_1)}{k} + \frac{(x_3 - x_1)}{k} + \ldots + \frac{(x_k - x_1)}{k}}_{> 0}$

$y_j = (\underbrace{x_1}_{< x_k} + \ldots + x_k) \frac{1}{k} < k x_k \cdot \frac{1}{k} = x_k < x_{k+1}$

$j = k+2$

$x_{j-k-1} = x_1 < y_j = (\underbrace{x_2 + \ldots + x_{k+1}}_{> (k-1) x_2}) \frac{1}{k}$

$y_j = (x_2 + \ldots + x_{k+1}) \frac{1}{k} < k x_{k+1} \frac{1}{k} = x_{k+1} < x_{k+2}$

Analog bis $j = m$

$j = m$

$x_{j-k-1} = x_{m-k-1} < y_j = (\underbrace{x_{m-k} + \ldots + x_{m-1}}_{> (k-1) x_{m-k}}) \frac{1}{k}$

$y_j = (\underbrace{x_{m-k}}_{< x_{m-1}} + \ldots + x_{m-1}) \frac{1}{k} < x_{m-1} < x_m$

$j = m+1$

$x_{j-k-1} = x_{m-k} < y_j = (\underbrace{x_{m+1-k} + \ldots + x_m}_{> (k-1) x_{m+1-k}}) \frac{1}{k}$

$y_j = (\underbrace{x_{m+1-k}}_{< x_m} + \ldots + x_m) \frac{1}{k} < x_m < x_{m+1}$

Analog bis $j = m+k$

$\Rightarrow$

$j = m+k$:

$$x_{j-k-1} = x_{m-1} < y_j = (x_m + \dots + x_{m+k-1})\frac{1}{k} \overset{*}{=} k\frac{1}{k}x_m = x_m$$

$$y_j = (x_m + \dots + x_{m+k-1})\frac{1}{k} \overset{*}{=} x_{m+k-1} \underset{\text{Vorr.}}{<} x_{m+k}$$

Damit ist $x_{j-k-1} < y_j < x_j \quad \forall j = 1, \dots, m+k$ gegeben

$\Rightarrow \det(A) \neq 0 \quad \Rightarrow$ Interpol. Problem eind. lösbar!

b) $$\frac{d}{dx} B_i^k(x) = \frac{d}{dx}\left(\frac{x - x_i}{x_{i+k+1} - x_i} B_i^{k-1}(x) + \frac{x_{i+k+1} - x}{x_{i+k+1} - x_i} B_{i+1}^{k-1}(x)\right)$$

$$B_i^0 = \begin{cases} 1/x_{i+1} - x_i & x \in [x_i, x_{i+1}] \\ 0 & \text{sonst} \end{cases}$$

$k = 0$: $\frac{d}{dx} B_i^0(x) = 0$

$k = 1$: $$\frac{d}{dx} B_i^1(x) = \frac{1}{x_{i+k+1} - x_i} B_i^0 + \frac{x - x_i}{x_{i+2} - x_i} \underbrace{\frac{d}{dx} B_i^0}_{=0}$$

$$+ \frac{x_{i+2} - x}{x_{i+2} - x_i} \underbrace{\frac{d}{dx} B_{i+1}^0}_{=0} + \frac{-1}{x_{i+2} - x_i} B_{i+1}^0$$

$$= \frac{1}{x_{i+2} - x_i}\left(B_i^0 - B_{i+1}^0\right)$$

$k = k+1$: $$\frac{d}{dx} B_i^{k+1}(x) = \frac{1}{x_{i+k+1} - x_i} B_i^k + \frac{x - x_i}{x_{i+k+1} - x_i} \frac{d}{dx} B_i^k$$

$$+ \frac{x_{i+k+1} - x}{x_{i+k+1} - x_i} \frac{d}{dx} B_{i+1}^k + \frac{-1}{x_{i+k+1} - x_i} B_{i+1}^k$$

$$= \frac{1}{x_{i+k+1} - x_i}\left(B_i^k - B_{i+1}^k\right) + \frac{x - x_i}{x_{i+k+1} - x_i} \frac{k}{x_{i+k+1} - x_i}\left(B_i^{k-1}(x) - B_{i+1}^{k-1}(x)\right)$$

$$+ \frac{x_{i+k+1} - x}{x_{i+k+1} - x_i} \frac{k}{x_{i+k+2} - x_{i+1}}\left(B_{i+1}^{k-1}(x) - B_{i+2}^{k-1}(x)\right)$$