

Aufgabe 1

$$a) \text{ Trapez-Regel: } I(f) = \frac{h}{2} \left(f(a) + f(b) + 2 \sum_{i=1}^{N-1} f(a+hi) \right)$$

$$h = \frac{1}{3}, a = 0, b = 2, N = \frac{2-0}{\frac{1}{3}} = 6$$

$$\Rightarrow I\left(\frac{1}{4+x^2}\right) = \left(\frac{1}{4} + \frac{1}{4+2^2} + 2 \left(\sum_{i=1}^5 \frac{1}{\left(\frac{1}{3} \cdot i\right)^2 + 4} \right) \right) \cdot \frac{1}{6}$$

$$= 0,39212$$

Simpson-Regel:

$$I(f) = \frac{h}{3} \left(\frac{1}{2} f(a) + \sum_{k=1}^{N-1} f(a+kh) + 2 \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} f\left(a+h\left(k-\frac{1}{2}\right)\right) + \frac{1}{2} f(b) \right)$$

$$\Rightarrow I\left(\frac{1}{4+x^2}\right) = \frac{1}{9} \left(\frac{1}{8} + \frac{1}{16} + \sum_{k=1}^5 \frac{1}{4+(kh)^2} + 2 \sum_{k=1}^6 \frac{1}{4+(h(k-\frac{1}{2}))^2} \right)$$

$$= 0,392699$$

$$\pi/8 \approx 0,392699$$

\Rightarrow Simpson-Regel geringfügig genauer, aber aufwändiger (2 Summen über $\approx N$ statt 1).

b) Fehlerabschätzung zusammengesetzte Trapez-Regel:

$$|E(f)| \leq \frac{(b-a)}{12} h^2 \max_{a \leq x \leq b} (|f''(x)|)$$

$$f(x) = \frac{1}{4+x^2}, f'(x) = \frac{-2x}{(4+x^2)^2}, f''(x) = \frac{-2(4+x^2)^2 + 2x \cdot 2(4+x^2) \cdot 2x}{(4+x^2)^4}$$

$$f'''(x) = \frac{12x(4+x^2)^3 - (-8+6x^2) \cdot 3(4+x^2)^2 \cdot 2x}{(4+x^2)^6} = \frac{-8+6x^2}{(4+x^2)^3}$$

$$= \frac{12x(4+x^2) - (-8+6x^2) \cdot 6}{(4+x^2)^6}$$

$$= \frac{(-96x+24x^3)(4+x^2)^2}{(4+x^2)^6} = \frac{24x(-4+x^2)}{(4+x^2)^4}$$

$$\text{Maxima: } f'''(x_0) \stackrel{!}{=} 0 = 24x(-4+x^2) \Rightarrow x = \pm 2, 0$$

Minima:

$$f''(2) = \frac{-8+6 \cdot 4}{(4+4)^3} = \frac{1}{32}$$

$$f''(-2) = \frac{-8+6 \cdot 4}{(4+4)^3} = \frac{1}{32}$$

$$f''(0) = \frac{-8}{4^3} = -\frac{1}{8} \leftarrow \text{Minimum}$$

$$\Rightarrow |E(f)| \leq \frac{2-0}{12} \cdot \frac{1}{9} \cdot \frac{1}{32} = 0,0005787$$

Aufgabe 2

a) $S \in S_{X,k} \Rightarrow x_0 < x_1 < \dots < x_m$

* $x_{-k+1} = x_{-k+2} = \dots = x_0 < x_1 < \dots < x_m = x_{m+1} = \dots = x_{m+k-1}$

Sei $A = (B_j(y_i))_{i=1, j=-k}^{m+k, m-1}$ (Matrix $(m+k) \times (m+k)$)

dabei $s(y_i) = \sum_{j=-k}^{m-1} \lambda_j B_j(y_i) = b_i$

Das Interpolationsproblem ist genau dann eindeutig lösbar, wenn $\det(A) \neq 0$!

Nach Schönberg/Whitney:

$\det(A) \neq 0 \Leftrightarrow x_{j-k-1} < y_j < x_j \quad \forall j = 1, \dots, m+k$

Für $j=1$:
 $x_{j-k-1} = x_{-k} \stackrel{\text{vorr.}}{=} x_{-k+1} = x_0 \stackrel{*}{=} x_0 < y_j = \left(\frac{x_0}{1-k} + \dots + x_0 \right) \frac{1}{k} = x_0$
 $x_j = x_0 < x_1$

$j=2$:
 $x_{j-k-1} = x_{1-k} \stackrel{*}{=} x_0 < y_j = \left(\frac{x_{2-k} + \dots + x_1}{k} \right) \frac{1}{k} = x_0 + \frac{x_1 - x_0}{k}$
 $y_j = \left(\frac{x_{2-k} + \dots + x_1}{k} \right) \frac{1}{k} < k x_1 \cdot \frac{1}{k} = x_1 < x_2$

Analog bis $x_{j-k-1} \leq x_0 \Leftrightarrow j = k+1$

$j = k+1$:
 $x_{j-k-1} = x_0 < y_j = \left(x_1 + \dots + x_k \right) \frac{1}{k} = k x_1 + \frac{(x_2 - x_1)}{k} + \frac{(x_3 - x_1)}{k} + \dots + \frac{(x_k - x_1)}{k}$
 $y_j = \left(\frac{x_1 + \dots + x_k}{k} \right) \frac{1}{k} < k x_k \cdot \frac{1}{k} = x_k < x_{k+1}$

$j = k+2$:
 $x_{j-k-1} = x_1 < y_j = \left(x_2 + \dots + x_{k+1} \right) \frac{1}{k} > (k-1)x_2$
 $y_j = \left(x_2 + \dots + x_{k+1} \right) \frac{1}{k} < k x_{k+1} \cdot \frac{1}{k} = x_{k+1} < x_{k+2}$

Analog bis $j = m$

$j = m$:
 $x_{j-k-1} = x_{m-k-1} < y_j = \left(x_{m-k} + \dots + x_{m-1} \right) \frac{1}{k} > (k-1)x_{m-k}$
 $y_j = \left(x_{m-k} + \dots + x_{m-1} \right) \frac{1}{k} < k x_{m-1} \cdot \frac{1}{k} = x_{m-1} < x_m$

$j = m+1$:
 $x_{j-k-1} = x_{m-k} < y_j = \left(x_{m+1-k} + \dots + x_m \right) \frac{1}{k} > (k-1)x_{m+1-k}$
 $y_j = \left(x_{m+1-k} + \dots + x_m \right) \frac{1}{k} < k x_m \cdot \frac{1}{k} = x_m < x_{m+1}$

Analog bis $j = m+k$

=)

$$j = m+k$$

$$x_{j-k-1} = x_{m-1} < y_j = (x_m + \dots + x_{m+k-1}) \frac{1}{k} \stackrel{*}{=} k \frac{1}{k} x_m = x_m$$

$$y_j = (x_m + \dots + x_{m+k-1}) \frac{1}{k} \stackrel{*}{=} x_{m+k-1} < \underset{\text{Vorr.}}{x_{m+k}}$$

Damit ist $x_{j-k-1} < y_j < x_j \quad \forall j = 1, \dots, m+k$ gegeben

$\Rightarrow \det(A) \neq 0 \quad \Rightarrow$ Interpol. Problem eind. lösbar!

$$b) \quad \frac{d}{dx} B_i^k(x) = \frac{d}{dx} \left(\frac{x-x_{i+k+1}}{x_{i+k+1}-x_i} B_i^{k-1}(x) + \frac{x_{i+k+1}-x}{x_{i+k+1}-x_i} B_{i+1}^{k-1}(x) \right)$$

$$B_i^0 = \begin{cases} \frac{1}{x_{i+1}-x_i} & x \in [x_i, x_{i+1}] \\ 0 & \text{sonst} \end{cases}$$

$$k=0: \quad \frac{d}{dx} B_i^0(x) = 0$$

$$\begin{aligned} k=1: \quad \frac{d}{dx} B_i^1(x) &= \frac{1}{x_{i+k+1}-x_i} B_i^0 + \frac{x-x_i}{x_{i+k+1}-x_i} \frac{d}{dx} B_i^0 \\ &\quad + \frac{x_{i+k+1}-x}{x_{i+k+1}-x_i} \frac{d}{dx} B_{i+1}^0 + \frac{-1}{x_{i+k+1}-x_i} B_{i+1}^0 \\ &= \frac{1}{x_{i+k+1}-x_i} (B_i^0 - B_{i+1}^0) \end{aligned}$$

$$\begin{aligned} k=k+1: \quad \frac{d}{dx} B_i^{k+1}(x) &= \frac{1}{x_{i+k+1}-x_i} B_i^k + \frac{x-x_i}{x_{i+k+1}-x_i} \frac{d}{dx} B_i^k \\ &\quad + \frac{x_{i+k+1}-x}{x_{i+k+1}-x_i} \frac{d}{dx} B_{i+1}^k + \frac{-1}{x_{i+k+1}-x_i} B_{i+1}^k \\ &= \frac{1}{x_{i+k+1}-x_i} (B_i^k - B_{i+1}^k) + \frac{x-x_i}{x_{i+k+1}-x_i} \frac{k}{x_{i+k+1}-x_i} (B_i^{k-1}(x) - B_{i+1}^{k-1}(x)) \\ &\quad + \frac{x_{i+k+1}-x}{x_{i+k+1}-x_i} \frac{k}{x_{i+k+1}-x_{i+1}} (B_{i+1}^{k-1}(x) - B_{i+2}^{k-1}(x)) \end{aligned}$$