

Besprechung zu Blatt 10 zu Analysis 3

Aufgabe 1

$$\begin{aligned}f_1(z) &= \Re(z) \\f_1(x+iy) &= \Re(x+iy) = x \quad (x, y \in \mathbb{R}) \\ \tilde{f}_1 : \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\(x, y) &\mapsto (\Re(f_1(x+iy)), \Im(f_1(x+iy))) \\f_1 \text{ holomorph} &\Leftrightarrow \tilde{f}_1 \text{ reell diffbar.} \\ J\tilde{f}_1(x_0, y_0) &= \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\tilde{f}_1(x, y) &= (x, 0) \Rightarrow J\tilde{f}_1(x_0, y_0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ \Rightarrow f_1 &\text{ nicht holomorph.}\end{aligned}$$

$$\begin{aligned}f_2(z) &= \Im(z), f_2(x+iy) = y, \tilde{f}_2(x, y) = (y, 0) \\ \Rightarrow J\tilde{f}_2(x_0, y_0) &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \Rightarrow f_2 &\text{ nicht holomorph}\end{aligned}$$

$$\begin{aligned}\lim_{w \rightarrow z_0} \frac{f(z_0) - f(w)}{z_0 - w} &= \lim_{w \rightarrow z_0} \frac{z_0^2 - w^2}{z_0 - w} \\ &= \lim_{w \rightarrow z_0} \frac{(z_0 - w)(z_0 + w)}{z_0 - w} = \lim_{w \rightarrow z_0} z_0 + w = 2z_0 \\ \Rightarrow f_3 &\text{ holomorph}\end{aligned}$$

$$\begin{aligned}f_4(z) &= |z|^2, f_4(x+iy) = |x+iy|^2 = \sqrt{x^2 + y^2}^2 = x^2 + y^2 \\ \tilde{f}_4(x, y) &= (x^2 + y^2, 0) \\ \Rightarrow J\tilde{f}_4(x_0, y_0) &= \begin{pmatrix} 2x_0 & 2y_0 \\ 0 & 0 \end{pmatrix} \\ \Rightarrow &\text{holomorph bei 0, aber nicht auf } \mathbb{C} \setminus \{0\}.\end{aligned}$$

$$\begin{aligned}f_5(x+iy) &= g(x+iy) + ig(x+iy), \quad g : \mathbb{C} \rightarrow \mathbb{R} \text{ reell diffbar.} \\ \tilde{f}_5(x, y) &= (g(x+iy), g(x+iy)) \\ J\tilde{f}_5(x_0, y_0) &= \begin{pmatrix} \frac{\partial g(x_0+iy_0)}{\partial x} & \frac{\partial g(x_0+iy_0)}{\partial y} \\ \frac{\partial g(x_0+iy_0)}{\partial x} & \frac{\partial g(x_0+iy_0)}{\partial y} \end{pmatrix} \\ \Leftrightarrow \frac{\partial g(x_0+iy_0)}{\partial x} &= \frac{\partial g(x_0+iy_0)}{\partial y} \wedge -\frac{\partial g(x_0+iy_0)}{\partial y} = \frac{\partial g(x_0+iy_0)}{\partial x} \\ \Rightarrow \frac{\partial g(x_0+iy_0)}{\partial y} &= -\frac{\partial g(x_0+iy_0)}{\partial y} \Rightarrow \frac{\partial g(x_0+iy_0)}{\partial y} = 0 \\ \Rightarrow \frac{\partial g(x_0+iy_0)}{\partial x} &= 0 \Rightarrow g \text{ konstant} \\ \Rightarrow f_5 &\text{ holomorph falls } g \text{ konst.}\end{aligned}$$

$$\begin{aligned}f_6(z) &= z^2 + 2z + \bar{z} \\ f_6(x+iy) &= (x+iy)^2 + 2(x+iy) + x-iy = (x^2 + 3x - y^2) + i(2xy + 2y - y) \\ \Rightarrow \tilde{f}_6(x, y) &= (x^2 + 3x + y^2, 2xy + y) \\ \Rightarrow J\tilde{f}_6(x_0, y_0) &= \begin{pmatrix} 2x_0 + 3 & -2y_0 \\ 2y_0 & 2x_0 + 1 \end{pmatrix} \\ \Rightarrow f_6 &\text{ nicht holomorph}\end{aligned}$$

Aufgabe 2

$$\begin{aligned}h : z &\mapsto (f(z))^2 \text{ holomorph} \Rightarrow f \text{ holomorph } g : z \mapsto z^2 \text{ holomorph nach Aufgabe 1} \\ h(z) &= (g \circ f)(z) \\ \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} &= J\tilde{h}(x_0, y_0) = J(g \circ f)(x_0, y_0) = J(\tilde{g} \circ \tilde{f})(x_0, y_0)\end{aligned}$$

$$\begin{aligned}
&= J\tilde{g}(\tilde{f}(x_0, y_0)) \circ J\tilde{f}(x_0, y_0) \\
&= \begin{pmatrix} \gamma & -\delta \\ \delta & \gamma \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a\gamma - c\delta & b\gamma - d\delta \\ a\delta + c\gamma & b\delta + d\gamma \end{pmatrix} \\
&\Rightarrow a\gamma - c\delta = b\delta + d\gamma \wedge -b\gamma + d\delta = a\delta + c\gamma \\
&\Rightarrow \underbrace{(a-d)}_{=0}\gamma + \underbrace{(-c-b)}_{=0}\delta = 0 \wedge \underbrace{(a-d)}_{=0}\delta + \underbrace{(c+b)}_{=0}\gamma = 0 \\
&\Rightarrow a-d=0 \wedge c+b=0 \\
&\Rightarrow a=d \wedge c=-b \\
&\Rightarrow J\tilde{f}(x_0, y_0) = \begin{pmatrix} a & -c \\ c & a \end{pmatrix} \\
&\Rightarrow f \text{ holomorph}
\end{aligned}$$

Aufgabe 3

Aufgabe 4

a) $\gamma(t) = e^{it}, t \in [0, 2\pi]$

$$\begin{aligned}
\int_{\gamma} |z| dz &= \int_0^{2\pi} |\gamma(t)| \dot{\gamma}(t) dt \\
&= \int_0^{2\pi} i \underbrace{|e^{it}|}_{=1} e^{it} dt = \int_0^{2\pi} i^{it} dt = \int_0^{2\pi} i(\cos(t) + i \sin(t)) dt \\
&= \int_0^{2\pi} -\sin(t) + i \cos(t) dt = \int_0^{2\pi} -\sin(t) dt + i \int_0^{2\pi} \cos(t) dt = 0 + i0 = 0
\end{aligned}$$

b) $\gamma_1(t) = 1 - i + 2it, t \in [0, 1]$

$$\gamma_2(t) = i\gamma_1(t)$$

$$\gamma_3(t) = i^2\gamma_1(t) = -\gamma_1(t)$$

$$\gamma_4(t) = i^3\gamma_1(t) = -i\gamma_1(t)$$

$$\gamma(t) = \begin{cases} \gamma_1(t) & , t \in [0, 1] \\ \gamma_2(t-1) & , t \in [1, 2] \\ \gamma_3(t-2) & , t \in [2, 3] \\ \gamma_4(t-3) & , t \in [3, 4] \end{cases}$$

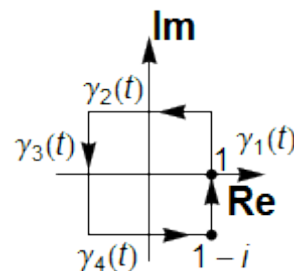
$$\gamma : [0, 4] \rightarrow \mathbb{C}$$

$$\int_{\gamma} \frac{1}{z} dz = \sum_{i=1}^4 \int_{\gamma_i} \frac{1}{z} dz$$

$$\int_{\gamma_i} \frac{1}{z} dz = \int_0^1 \frac{1}{\gamma_i(t)} \dot{\gamma}_i(t) dt = \int_0^1 \frac{1}{\gamma_1(t)} \dot{\gamma}_1(t) dt$$

(da für $i=2,3,4$ sich der konst. Vorfaktor wegekürzt)

$$\begin{aligned}
\int_{\gamma} \frac{1}{z} dz &= 4 \int_{\gamma_1} \frac{1}{z} dz = 4 \int_0^1 \frac{1}{1-i+2it} \cdot (2i) dt \\
&= 8i \int_0^1 \frac{1}{1+i(-1+2t)} dt = 8i \int_0^1 \frac{1-i(-1+2t)}{1^2+(-1+2t)^2} dt \\
&= 8i \int_0^1 \frac{1+i(1-2t)}{1+(1-2t)^2} dt \\
&= 8i \int_0^1 \frac{1}{1+(1-2t)^2} dt + 8i^2 \int_0^1 \frac{1-2t}{1+(1-2t)^2} dt \\
&= -4i \int_0^1 \frac{-2}{1+(1-2t)^2} + 2 \int_0^1 \frac{-4(1-2t)}{1+(1-2t)^2} \\
&= -4i[\arctan(1-2t)]_0^1 + 2[\ln(1+(1-2t)^2)]_0^1
\end{aligned}$$



$$\begin{aligned}
& -4i(\arctan(-1) - \arctan(1)) + 2 \underbrace{(\ln(2) - \ln(2))}_{=0} \\
& = -4i(-\arctan(1) - \arctan(1)) = 8i \arctan(1) \\
& = 8i \frac{\pi}{4} = 2\pi i
\end{aligned}$$

oder:

$$\begin{aligned}
\gamma & \equiv \tilde{\gamma}, \quad \tilde{\gamma}(t) = e^{2\pi i t}, \quad t \in [0, 1] \\
\int_{\gamma} \frac{1}{z} dz & = \int_{\tilde{\gamma}} \frac{1}{z} dz = \int_0^1 \frac{1}{e^{2\pi i t}} 2\pi i e^{2\pi i t} dt
\end{aligned}$$

c) $\gamma_1(t) = 1 + it$
 $\gamma_2(t) = 1 + i - t$
 $\gamma_3(t) = i - it$
 $\gamma_4(t) = t$
 γ wie oben

$$\int_{\gamma} |z|^2 dz$$

(nicht holomorph! \Rightarrow Homotopiesatz nicht anwendbar!)

$$\int_{\gamma} |z|^2 dz = \int_{\gamma_1} |z|^2 dz + \dots + \int_{\gamma_4} |z|^2 dz$$

$$\int_{\gamma_1} |z|^2 dz = \int_0^1 |1 + it|^2 i dt = \int_0^1 (1 + t^2) i dt = i \left[t + \frac{1}{3} t^3 \right]_0^1 = \frac{4}{3} i$$

$$\int_{\gamma_2} |z|^2 dz = \int_0^1 |1 - t + i|^2 (-1) dt = - \int_0^1 (1 - t)^2 + 1 dt = -\frac{4}{3}$$

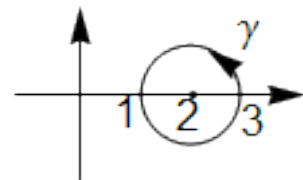
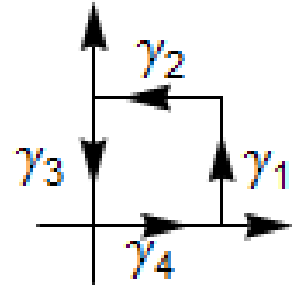
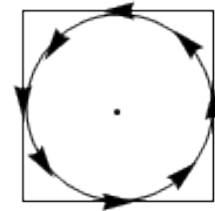
$$\int_{\gamma_3} |z|^2 dz = \int_0^1 |i - it|^2 (-i) dt = -\frac{1}{3} i$$

$$\int_{\gamma_4} |z|^2 dz = \int_0^1 t^2 dt = \frac{1}{3}$$

$$\Rightarrow \int_{\gamma} |z|^2 dz = \frac{4}{3} i - \frac{4}{3} - \frac{1}{3} i + \frac{1}{3} = i - 1$$

d) $\frac{1}{z}$ holomorph γ nullhomotop

$$\Rightarrow \text{mit Homot.Satz: } \int_{\gamma} \frac{1}{z} dz = 0$$



Aufgabe 5

$$\int \underbrace{\langle i\bar{f}(\gamma(t)), \dot{\gamma}(t) \rangle}_{\in \mathbb{R}} dt = i \int \underbrace{\langle \bar{f}(\gamma(t)), \dot{\gamma}(t) \rangle}_{\substack{\in \mathbb{R} \\ \in \mathbb{C} \setminus (\mathbb{R} \setminus \{0\})}} dt$$

$\in \mathbb{R}, = \left\langle \begin{pmatrix} v \\ u \end{pmatrix}, \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \right\rangle$