Theo 4 Übungs-Tutorium

Mitgeschrieben und geLATEXt von Julian Bergmann

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1 Tutorium vom 29.04.2011 (Blatt 1)

1.1 Hausaufgabe 1

Compton-Wellenlänge eines Teilchen mit Masse m: $\lambda_c = \frac{h}{mc}$

$h = 6.626 \cdot 10^{-34} Js = 4.1357 \cdot 10^{-15} eVs$

$\hbar = \frac{h}{2\pi} = 1.0546 \cdot 10^{-34} Js = 6.582 \cdot 10^{-16} eVs$

$\hbar c = 197.33 MeV fm$

$1 \frac{MeV}{c^2} = 1.79 \cdot 10^{-30} kg$

a) $m_e = 9.1 \cdot 10^{-34} kg = 0.51 \frac{MeV}{c^2}$

$\lambda_c(e^-) = \frac{2\pi\hbar}{m_e c} = \frac{2\pi\hbar c}{m_e c^2} = \frac{2 \cdot 3.14 \cdot 197.33 MeV fm}{0.51 MeV}$

$\approx 2429.86745 fm \approx 2.43 \cdot 10^{-12} m$

b,c) analog

d) $m(Auto) = t = 10^3 kg = 0.56 \cdot 10^{33} \frac{MeV}{c^2}$

$\lambda_c(Auto) = \frac{2\pi\hbar}{m(Auto)c} = \frac{2\pi\hbar c}{mc^2} = \frac{2 \cdot 3.14 \cdot 197.33 MeV fm}{0.56 \cdot 10^{33} MeV}$

$\approx 22.13 \cdot 10^{-31} fm \approx 22.13 \cdot 10^{-46} m$

$\lambda_{deBroglie} = \frac{h}{p} = \frac{h}{mv} ...$

1.2 Hausaufgabe 2

$\hbar = 10^{-3} Js$ nur in Quantum-Land

$d = 20cm = \Delta, \quad M = 0.1g$

$\boxed{\Delta x \cdot \Delta p \geq \frac{\hbar}{2}}$

$\Rightarrow \Delta P = \frac{\hbar}{2\pi} = \frac{10^{-3} kb \frac{m}{s^2} ms}{2 \cdot 0.2m} = 2.5 \cdot 10^{-3} kg \frac{m}{s}$

$\Delta\sigma = \frac{\Delta p}{M} = \frac{2.5 \cdot 10^{-3} kg \frac{m}{s}}{10^{-4} kg} = 25 \frac{m}{s} = 90 \frac{km}{h}$

$\Delta v_{min} \geq 90 \frac{km}{h}$

d.h. Die Kerne besitzen min. $90 \frac{km}{h}$!

1.3 Hausaufgabe 3

a) **Lagrangefunktion**:

$L = T - V = \frac{1}{2} m v^2 - \frac{e^2}{r}, \quad v^2 = \dot{\vec{r}}^2$

$\vec{r}(t) = (r(t)\cos(\phi(t)), r(t)\sin(\phi(t)))$

$\dot{\vec{r}} = (\dot{r}\cos(\varphi) + r(-\sin(\phi))\dot{\phi}, \dot{r}\sin(\phi) + r\cos(\phi)\dot{\phi})$

$\Rightarrow \dot{\vec{r}}^2 = \dot{r}^2 + r^2\dot{\phi}^2 = v^2$

Hamiltonfunktion:

$H = T + V = \frac{1}{2}mv^2\frac{e^2}{r} = \frac{1}{2}m\dot{r} + \frac{1}{2}mr^2\dot{\phi}^2 - \frac{e^2}{r}$

Kanonische Größen:

$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}, \quad p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$

$\Rightarrow \frac{p_r^2}{2m} = \frac{m^2\dot{r}^2}{2m} = \frac{1}{2}m\dot{r}^2, \quad \frac{p_\phi^2}{2m} = \frac{m^2r^4\dot{\phi}^2}{2mr^2} = \frac{1}{2}mr^2\dot{\phi}^2$

$H = T + V = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} - \frac{e^2}{r}$

$\frac{\partial L}{\partial \phi} = 0, \quad \frac{\partial L}{\partial \dot{\phi}} \neq 0 \Rightarrow \phi$ zyklisch.

Konstanten der Bewegung:

(1) $H = E$, da $H(q,p)$ nicht explizit zeitabhängig.

(2) $p_\phi = L$, da ϕ zyklische Variable.

L: konstanter Drehimpuls.

Hamilton'sche Gleichungen:

$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0, \quad \dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2}$

$\dot{p}_r = -\frac{\partial H}{\partial r} = -\frac{e^2}{r^2} + \frac{p_\phi^2}{mr^3}0, \quad \dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}$

b) Mit $p_\phi = L$:

$E = \frac{p_r^2}{2m} + \frac{L^2}{2mr^2} - \frac{e^2}{r}$

$\Rightarrow p_r = \frac{\sqrt{2m(Er^2+e^2r)-L^2}}{r}$

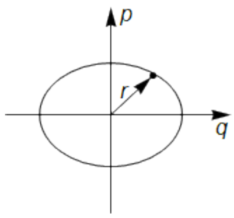
c) Umkehrpunkte: $p_r = 0$

$2m(Er^2 + e^2r) - L^2 = 0 \;\Rightarrow\; r^2 + \frac{e^2}{E}r - \frac{L^2}{2mE} = 0$

$\Rightarrow r_{1,2} = -\frac{e^2}{2E} \pm \frac{\sqrt{\Delta}}{4mE}$ mit $\Delta = -4m(2EL^2 + me^4)$

d,e) wegen Einschüben fallen gelassen.

1.4 Einschub: harm. Oszillator



$E = T + V = \frac{m\dot{q}^2}{2} + \frac{fq^2}{2} = \frac{p^2}{2m} + \frac{fq^2}{2}$

$\frac{p^2}{2mE} + \frac{q^2}{2mE} = 1, \quad a = \sqrt{2mE}, \; b = \sqrt{\frac{2E}{f}}$

$\frac{p^2}{a^2} + \frac{q^2}{b^2} = 1 \;\Rightarrow$ Ellipse

$F = \int f(x)dx$, Ellipse: $\oint pdq = \pi ab = \pi\sqrt{2mE}\sqrt{\frac{2E}{f}} = 2\pi E\sqrt{\frac{m}{f}}$

$\nu = \frac{1}{2\pi}\sqrt{\frac{f}{m}} \quad\Rightarrow\quad \oint pdq = \pi ab = 2\pi E\sqrt{\frac{m}{f}} = \frac{E}{\nu} = nh$

Oszillator-Zustände: $E_n = nh\nu$

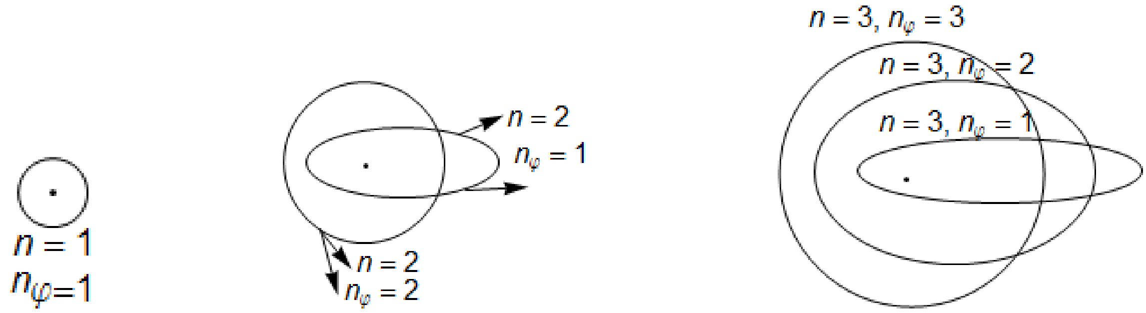
$\Rightarrow \boxed{\oint pdq = nh}$

Hinweis zur Schreibweise: f=D=k

1.5 Bohr'sche Bahnen

$n = n_r + n_\varphi, \quad n_\varphi = 1, ..., n$

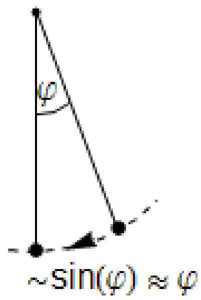
Bahnen: a) b) c)



$$\boxed{E_n = -\frac{me^4}{2n^2\hbar^2}}$$

2 Tutorium vom 06.05.11 (Blatt 2)

2.1 Hausaufgabe 3



$\ddot{x} + \omega_0^2 \sin(x) = F\cos(\omega t)$

a) $\sin(x) = \sum\limits_k (-1)^k \frac{x^{2k+1}}{(2k+1)!} \approx x - \frac{1}{6}x^3$

$\Rightarrow \ \ddot{x} + \omega_0 x - \frac{1}{6}\omega_0 x^3 = F\cos(\omega t)$

b) $x' = \frac{dx}{d\tau}, \quad \dot{x} = \frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} = x'\omega$

$\Rightarrow \ddot{x} = \omega^2 x''$

$\Rightarrow x'' + \Omega^2 x - \frac{1}{6}\Omega^2 x^3 = \Gamma\cos(\tau)$

c) $(x_0'' + \varepsilon_1'' + ...) + \Omega^2(x_0 + \varepsilon_1 + ...) - \varepsilon(x_0 + \varepsilon x_1 + ...)^3 = \Gamma\cos(\tau)$

$x_0'' + \Omega^2 x_0 = \Gamma\cos(\tau)$

$\Rightarrow \text{DGLs}: \ x_1'' + \Omega^2 x_1 = x_0^3$

$x_2'' + \Omega^2 x_2 = 3x_0^2 x_1$

d) 2π-periodisch nur für $a_0 = b_0 = 0$

e) $x_0(\tau) = \frac{\Gamma}{\Omega^2-1}\cos(\tau)$ einsetzen.

$$x_1'' + \Omega^2 x_1 = \frac{\Gamma^3}{(\Omega^2-1)^3}\underbrace{\left(\frac{3}{4}\cos(\tau) + \frac{1}{4}\cos(3\tau)\right)}_{\cos^3(\tau)}$$

f) $x_1(\tau) = A\cos(\tau) + B\cos(3\tau) \Rightarrow x_1''(\tau) = -A\cos(\tau) - 9B\cos(3\tau)$

$x_1'' + \Omega^2 x_1 = A(\Omega^2-1)\cos(\tau) + B(\Omega^2-9)\cos(3\tau) \stackrel{!}{=} \frac{3}{4}\frac{\Gamma^3}{(\Omega^2-1)^3}\cos(\tau) + \frac{1}{4}\frac{\Gamma^3}{(\Omega^2-1)^3}\cos(3\tau)$

$\Rightarrow A = \frac{3}{4}\frac{\Gamma^3}{(\Omega^2-1)^4}, \quad B = \frac{1}{4}\frac{\Gamma^3}{(\Omega^2-1)^3(\Omega^2-9)}$

g) $x(\varepsilon,\tau) = \frac{\Gamma}{\Omega^2-1}\cos(\tau) + \varepsilon\left(\frac{3}{4}\frac{\Gamma^3}{(\Omega^2-1)^4}\cos(\tau) + \frac{1}{4}\frac{\Gamma^3}{(\Omega^2-1)^3(\Omega^2-9)}\cos(3\tau)\right) + O(\varepsilon^2)$

2.2 Hausaufgabe 4

a) $\Psi(x,0) = \int\limits_{-\infty}^{\infty} f(p)\Psi_p(x-x_0,0)dp$

$= \int\limits_{-\infty}^{\infty} \underbrace{\frac{1}{(2\pi)^{1/4}\sqrt{\sigma_p}}}_{c_1} e^{-\frac{(p-p_0)^2}{4\sigma_p^2}} \underbrace{\frac{1}{(2\pi\hbar)^{1/2}}}_{c_2} e^{-\frac{i}{\hbar}(0-p(x-x_0))}dp$

Integrand:

$e^{-\frac{(p-p_0)^2}{4\sigma_p^2}} e^{\frac{i}{\hbar}p(x-x_0)} = e^{\frac{-p^2-2pp_0+p_0^2}{4\sigma_p^2}} e^{\frac{i}{\hbar}p(x-x_0)}$

$= e^{-\frac{p^2}{4\sigma_p^2}+\frac{2p_0 p}{4\sigma_p^2}+\frac{i}{\hbar}(x-x_0)p} e^{-\frac{p_0^2}{4\sigma_p^2}}$

$= exp(\underbrace{-\frac{1}{4\sigma_p^2}}_{a} p^2 + \underbrace{[\frac{p_0}{2\sigma_p^2} + \frac{i}{\hbar}(x-x_0)]}_{b} p + \underbrace{(-\frac{p_0^2}{4\sigma_p^2})}_{c})$

$\Rightarrow ax^2 + bx + c = a(x^2 + \frac{b}{a}x + (\frac{b}{2a})^2 - (\frac{b}{2a})^2) + c$

$\Psi(x,0) = c_1, c_2 \int\limits_{-\infty}^{\infty} e^{a(p+\frac{b}{2a})^2} e^{c-\frac{b^2}{4a}} dp$

$= c_1 c_2 e^{c-\frac{b^2}{4a}} \underbrace{\int\limits_{-\infty}^{\infty} e^{-\alpha(p-\beta)^2} dp}_{=\sqrt{\frac{\pi}{\alpha}}}, \quad \text{mit } \alpha = \frac{1}{4\sigma_p^2}$

$= c_1 c_2 e^{c-\frac{b^2}{4a}} \cdot 2\sigma_p\sqrt{\pi}$

Exponent:

$x - \frac{b^2}{4a} = -\frac{p_0^2}{4\sigma_p^2} - \frac{1}{4(-\frac{1}{4\sigma_p})}[\frac{p_0}{2\sigma_p^2} + \frac{i}{\hbar}(x-x_0]^2$

$= -\frac{p_0^2}{4\sigma_p^2} + \sigma_p^2[\frac{p_0^2}{4\sigma_p^4} + i\frac{p_0(x-x_0)}{\sigma_p^2\hbar} - \frac{(x-x_0)^2}{\hbar}]$

$= -\frac{(x-x_0)^2\sigma_p^2}{\hbar^2} + \frac{i}{\hbar}p_0(x-x_0)$

Damit: $\Psi(x,0) = \underbrace{\frac{1}{(2\pi)^{1/4}\sqrt{\sigma_p}} \cdot \frac{1}{(2\pi\hbar)^2} \cdot 2\sigma_p\sqrt{\pi}}_{\frac{1}{(2\pi)^{1/4}}\sqrt{\frac{2\sigma_p}{\hbar}}} e^{-\frac{\sigma_p^2}{\hbar^2}(x-x_0)^2} \cdot e^{\frac{i}{\hbar}p_0(x-x_0)}$

Für t=0: $\sigma_p\sigma_x = \frac{\hbar}{2} \begin{cases} \Rightarrow \sqrt{\frac{2\sigma_p}{\hbar}} = \frac{1}{\sqrt{\sigma_x}} \\ \Rightarrow \frac{\sigma_p^2}{\hbar^2} = \frac{1}{4\sigma_x^2} \end{cases}$

$\Rightarrow \Psi(x,0) = \frac{1}{(2\pi)^{1/4}\sqrt{\sigma_x}} e^{-\frac{(x-x_0)^2}{4\sigma_x}} e^{\frac{i}{\hbar}p_0(x-x_0)} = M(x,0)e^{i\phi(x,0)}$

b) $\int\limits_{-\infty}^{\infty} |\Psi(x,0)|^2 dx = \frac{1}{\sqrt{2\pi}\sigma_x} \int\limits_{-\infty}^{\infty} e^{\frac{(x-x_0)^2}{2\sigma_x}} dx$

$= \frac{1}{\sqrt{2\pi}\sigma_x}\sqrt{\frac{\pi}{\frac{1}{2\pi\sigma_x^2}}} = \frac{1}{\sqrt{2\pi}\sigma_x} \cdot \sqrt{2\pi}\sigma_x = 1$

3 Tutorium vom 13.05.11 (Blatt 3)

3.1 Hausaufgabe 6

$\Psi(x) = \mathcal{N} \exp(-\frac{(x-x_0)^2}{2\sigma^2})$

a) Substitution: $x - x_0 \to x$

$\int\limits_{-\infty}^{\infty} \Psi^2(x)dx = \mathcal{N}^2 \int\limits_{-\infty}^{\infty} e^{-\frac{x^2}{\sigma^2}} dx = \mathcal{N}\sqrt{\pi}\sigma$

$\Rightarrow \mathcal{N} = (\pi\sigma^2)^{-\frac{1}{4}}$

b) Nach Vorliebe von Herrn Prof. Dr. Lenske:

$\Psi(p) = \int\limits_{-\infty}^{\infty} e^{+ipx}\Psi(x)dx$

$\Psi(x) = \int\limits_{-\infty}^{\infty} e^{-ipx}\Psi(p)\frac{dp}{2\pi}$

$\int\limits_{-\infty}^{\infty} e^{-\alpha(x-\beta)^2}dx = \sqrt{\frac{\pi}{\alpha}}$

$\Psi(x) = \mathcal{N} \int\limits_{-\infty}^{\infty} dx e^{ipx - \frac{(x-x_0)^2}{2\sigma^2}} = \mathcal{N} \int\limits_{-\infty}^{\infty}$

$a = -\frac{(x-x_0)^2}{2\sigma^2} + ipx = -\frac{x^2}{2\sigma^2} + \frac{x_0 x}{\sigma^2} - \frac{x_0^2}{2\sigma^2} + ipx$

$= -\frac{1}{2\sigma^2}(x^2 - (2x_0 + ip \cdot 2\sigma^2)x + x_0^2)$

$= -\frac{1}{2\sigma^2}((\underbrace{x + (x_0 + ip\sigma^2)^2}_{\tilde{x}})^2 - (x_0 + ip\sigma^2)^2 + x_0 2)$

$= -\frac{1}{2\sigma^2}(\tilde{x}^2 - 2ip\sigma x_0 + p^2\sigma^4)$

Das Gauß-Integral über $\tilde{x}$ liefert den Faktor $\sqrt{2\pi}\sigma$ und man erhält:

$\Psi(p) = (4\pi\sigma^2)^{\frac{1}{4}} e^{-\frac{\sigma^2 p^2}{2} + ipx_0}$

Die Fourier-Transformation ist norm-erhaltend: $\int\limits_{-\infty}^{\infty} \frac{dp}{2\pi}|\Psi(p)|^2 = 1$

c) $x - x_0 \to u$ n=1:

$$< x >= \int\limits_{-\infty}^{\infty} dx\Psi^*(x)x\Psi(x) = \mathcal{N} \int\limits_{-\infty}^{\infty} du(u + x_0)e^{-\frac{u^2}{\sigma^2}}$$

$$= \frac{1}{\sqrt{\pi\sigma}} = x_0\sqrt{\sigma} = x_0, \text{ weil } \int\limits_{-\infty}^{\infty} dxxe^{-\frac{x^2}{\sigma^2}} = 0$$

$$< p >= 0$$

n=2:

$$< x^2 >= \int_{-\infty}^{\infty} dx\Psi^*(x)x^2\Psi(x) = \mathcal{N}^2 \int_{-\infty}^{\infty} dx' \underbrace{(x+x_0)^2}_{x^2+2x_0x+x_0^2} e^{-\frac{x'^2}{\sigma^2}}$$

$$= \mathcal{N}^2 \left(\int_{-\infty}^{\infty} dx x^2 e^{-\frac{x^2}{\sigma^2}} + 2x_0 \underbrace{\int_{-\infty}^{\infty} dx x e^{-\frac{x^2}{\sigma^2}}}_{=0,\ ungerd.\ Integrant} + \int_{-\infty}^{\infty} dx x_0^2 e^{-\frac{x^2}{\sigma^2}} \right)$$

$$= \frac{1}{\sqrt{\pi}\sigma}(\frac{1}{2}\sqrt{\pi}\sigma\sqrt{\sigma} + x_0^2\sqrt{\pi}\sigma) = \frac{1}{2}\sqrt{\sigma} + x_0^2$$

$$\Psi(p) = (4\pi\sigma^2)^{\frac{1}{4}} e^{-\frac{\sigma^2 p^2}{2} + ipx_0}$$

$$< p^2 >= \int_{-\infty}^{\infty} \frac{dp}{2\pi}\Psi^*(p)p^2\Psi(p)$$

$$= (4\pi\sigma^2) \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{-\sigma^2 p^2} p^2$$

$$= 2\sigma\sqrt{pi}\frac{1}{2\pi}\frac{\sqrt{\pi}}{2(\sigma^2)^{\frac{3}{2}}} = \frac{1}{2\sqrt{\sigma}}$$

e) $(\Delta x)^2 =< x^2 > - < x >^2 = kr2\sqrt{\sigma} + x_0^2 - x_0^2 = \frac{1}{2}\sqrt{\sigma}$

$(\Delta p)^2 =< p^2 > - < p >^2 = \frac{1}{2\sqrt{\sigma}}$

$\Delta x \cdot \Delta p = (\frac{1}{2\sqrt{\sigma}} \cdot \frac{1}{2}\sqrt{\sigma})^{\frac{1}{2}} = \frac{1}{2}$

$\Rightarrow \Delta x \Delta p \geq \frac{\hbar}{2}$ mit $\hbar = 1$

d) 1. n=1:$[x,p] = i\hbar, \quad [A,BC] = [A,B]C + B[A,C]$

$[x,p^n] = i\hbar n p^{n-1}$

$[x,p^{n+1}] = [x,p^n]p + p^n[x,p] = i\hbar n p^n + i\hbar p^n = i\hbar(n+1)p^{(n+1)-1}$

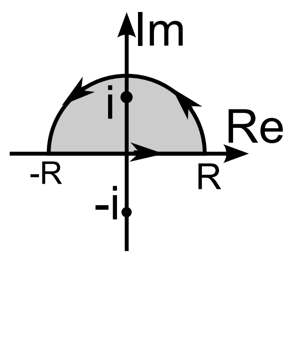
2. $[p, f(x)] = \alpha f'(x)$

Taylor: $f(x) = \sum_n \frac{1}{n!} f^{(n)}(0)x^n, \quad f'(x) = \sum_n \frac{1}{n!}(n f^{(n)}(0)x^{n-1})$

$[p, f(x)] = \sum_n \frac{1}{n!} f^{(n)}(0)[p, x^n] = \sum_n \frac{1}{n!} f^{(n)}(0)(-i\hbar n x^{n-1}) = -i\hbar f'(x)$

Auch über Anwendung auf $\Psi(x)$ und Ableitung.

3.2 Zusatzaufgabe



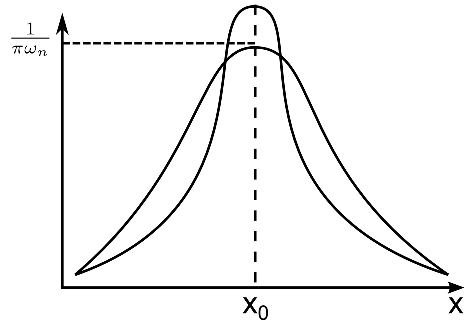
$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \arctan\big|_{-\infty}^{\infty} = \pi$$

Residuen-Satz:

$$\int_{-\infty}^{\infty} \frac{dz}{z^2+1}$$ Nennernullstellen: $z = \pm i$

$$= 2\pi i \sum_\alpha Res_\alpha f(z) = 2\pi i \cdot Res_{z=i} f = 2\pi i \frac{g(z)}{h'(u)}\Big|_{z=i} = 2\pi i \frac{1}{2i} = \pi$$

3.3 Hausaufgabe 7



$ln(x) = A\frac{1}{(x-x_0)^2+\omega_n^2}, \quad \omega_n = \frac{\omega_0}{n}, \quad n \in \mathbb{N}$

a) $A\int\limits_{-\infty}^{\infty} \frac{1}{(x-x_0)^2+\omega_n^2}dx = \frac{A}{\omega_n^2}\int\limits_{-\infty}^{\infty} \frac{1}{(\frac{(x-x_0)^2}{\omega_n})+1}$

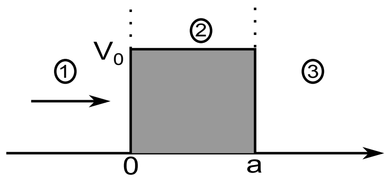
Substution: $(\frac{x-x_0}{\omega_n})^2 = y^2 \Rightarrow dy = \frac{dx}{\omega_n}$

$\frac{A}{\omega_n^2}\int\limits_{-\infty}^{\infty} \frac{1}{1+y^2}\omega_n dy = \frac{A}{\omega_n}\arctan(y)\Big|_{-\infty}^{\infty} = \frac{\pi A}{\omega_n} \stackrel{!}{=} 1$

$\Rightarrow A = \frac{\omega_n}{\pi}$

4 Tutorium vom 20.05.11 (Blatt 4)

4.1 Hausaufgabe 8



a) $E < V_0, k^2 = \frac{2m}{\hbar^2}E, \quad \kappa = \frac{2m}{\hbar^2}(V_0 - E)$

$\psi_I(x) = A_1e^{ikx} + B_1e^{-ikx}$ für 1

$\psi_2(x) = A_2e^{-\kappa x} + B_2e^{\kappa x}$ für 2

$\psi_3(x) = A_3e^{ikx}$ für 3

($B_3 = 0$ da Teilchen von links ankommt, also kein Wellenteilchen von rechts.)

Anschlussbedingungen:

$\psi_1(0) = \psi_2(0); \quad \psi_2(a) = \psi_3(a)$

$\psi_1'(0) = \psi_2'(0); \quad \psi_2'(a) = \psi_3'(a)$

$\psi_1'(x) = A_1ike^{ikx} - B_1ike^{-ikx}$

$\psi_2'(x) = -A_2\kappa e^{-\kappa x} - B_2\kappa e^{\kappa x}$

$\psi_3'(x) = A_3ike^{ikx}$

$(1): \ A_1 + B_1 = A_2 + B_2$

$(2): \ A_2e^{-\kappa a} + B_2e^{\kappa a} = A_3e^{ika}$

$(3): \ A_1ik - B_1ik = -A_2\kappa + B_2\kappa$

$(4): \ -A_2\kappa e^{-\kappa a} + B_2\kappa e^{\kappa a} = A_3ike^{ika}$

$(4) \Rightarrow B_2e^{\kappa a} = \frac{A_3ke^{ika}+A_2\kappa e^{-\kappa a}}{\kappa}$

Einsetzen in (2): $A_2e^{-\kappa a} + \frac{A_3ke^{ika}+A_2\kappa e^{-\kappa a}}{\kappa} = A_3e^{ika}$

$\Rightarrow A_2(e^{-\kappa a} + e^{-\kappa a}) = A_3(e^{ika} - \frac{ik}{\kappa}e^{ika}$

$\Rightarrow A_2 = \frac{e^{ika}-ike^{ika}/\kappa}{2e^{-\kappa a}}A_3 = \boxed{A_2 = \frac{1}{2}e^{ika}e^{\kappa a}(1-\frac{1}{\lambda})A_3} \quad (\lambda = \frac{\kappa}{k})$

A_2 einsetzen in (2):

$\boxed{B_2 = \frac{1}{2}e^{ika}e^{-\kappa a}(1+\frac{1}{\lambda})A_3}$

$\boxed{B_1 = -\frac{i}{2}e^{ika}(\lambda+\frac{1}{\lambda})\sinh(\kappa a)A_3}$

$(1) \Rightarrow A_1 = A_2 + B_2 - B_1$

Für A_2, B_2 und B_1 einsetzen

$\Rightarrow \boxed{A_1 = e^{ika}(\cosh(\kappa a) + \frac{i}{2}(\lambda - \frac{1}{\lambda})\sinh(\kappa a))A_3}$

b) $T = |\frac{A_3}{A_1}|^2, \quad R = |\frac{B_1}{A_1}|^2$

$T = \frac{1}{1+\frac{1}{4}(\lambda+\frac{1}{\lambda})^2\sinh^2(\kappa a)}$

$R = \frac{\frac{1}{4}(\lambda+\frac{1}{\lambda})^2\sinh^2(\kappa a)}{1+\frac{1}{4}(\lambda+\frac{1}{\lambda})^2\sinh^2(\kappa a)}$

$T + R = 1$

c) klassische Mechanik: für $E < V_0: \quad T = 0, R = 1$

Sehr breite Barriere: $\kappa a >> 1: \quad \lim_{x\to\infty}\sinh(x) = \infty$

$\Rightarrow T \to 0$ für $\kappa a \to \infty$

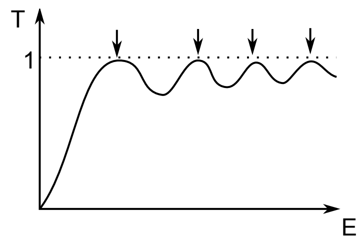
d) klassische Mechanik für $E \geq V_0: \quad T = 1, R = 0$

Für $E > V_0$ wird κ imaginär

Setze daher $\kappa = -iK, \quad \lambda = \frac{\kappa}{K} = -i\frac{K}{k} = -i\Lambda$

Gl lauten dann: $T = \frac{1}{1+\frac{1}{4}(\Lambda-\frac{1}{\Lambda})^2\sin^2(Ka)}$

$R = \frac{\frac{1}{4}(\Gamma-\frac{1}{\Gamma})^2\sin^2(Ka)}{1+\frac{1}{4}(\Lambda-\frac{1}{\Lambda})^2\sin^2(Ka)}$



e) Transmission wir 1, wenn $\sin(Ka) = 0$, d.h. $\boxed{Ka = n\pi}$

4.2 Hausaufgabe 9

a) $V(x) = \begin{cases} \infty & , x < -a \\ 0 & , -a \leq x < 0 \\ V_0\delta(x) & , x = 0 \\ 0 & , 0 < x \leq a \\ \infty & , x > a \end{cases}$

Potential symm. um 0: Lösungen haben definierte Parität

b) $\left.\begin{array}{l} \text{Region 1: } -a < x < 0 \\ \text{Region 2: } 0 < x < a \end{array}\right\}$ jeweils $V(x) = 0$

$-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} = E\Psi$

$\frac{d^2\Psi}{dx^2} = -k^2\Psi mitk^2 = \frac{2mE}{\hbar^2}$

c) Ansätze:

$\psi_1(x) = A_1 \sin(kx) + B_1 \cos(ka)\psi_2(x) = A_2 \sin(kx) + B_2 \cos(ka)$

Paritätsbed.:

gerade: $\psi(-x) = \psi(x) \Rightarrow \psi_1(-x) = \psi_2(x)$

$\forall - A_1 \sin(kx) + B_1 \cos(kx) = A_2 \sin(kx) + B_2 \cos(kx) \Rightarrow B_1 = B_2, \quad A_1 = -A_2$

ungerade: $\Rightarrow A_1 = A_2, \quad B_1 = -B_2$

Stetigkeit bei $x = 0: \quad \psi_1(0^-) = \psi_2(0^+)$

$A_1 \sin(0) + B_1 \cos(0) = A_2 \sin(0) + B_2 \cos(0) \Rightarrow B1 = B_2$

Für ungerade Parität gilt auch $B_1 = -B_2 \Rightarrow B_1 = B_2 = 0$

Unstetigkeitssprung: der ersten Ableitung:

Es gilt $\frac{-\hbar^2}{2m}(\psi_2'(0^+) - \psi_1'(0^-)) + V_0\Psi(0) = 0$

$\psi_1'(x) = A_1 k \cos(kx) - B_1 k \sin(kx)$

$\psi_2'(x) = A_2 k \cos(kx) - B_2 k \sin(kx)$

$\varepsilon \to 0^+:$

$\frac{-\hbar^2}{2m}(A_2 k \cos(k\varepsilon) - B_2 k \sin(k\varepsilon)) - (A_1 k \cos(k\varepsilon) - B_1 k \sin(k\varepsilon))) + V_0 B_1 = 0$

$\Rightarrow \frac{-\hbar^2}{2m}(A_2 k - A_1 k) + V_0 B_1 = 0$ (für gerade Parität)

gerade Par.: $A_1 = -A_2 \;\; \Rightarrow \;\; -\frac{\hbar^2}{2m} 2a_2 k + V_0 B_1 = 0 \Rightarrow A_2 = \frac{mV_0}{\hbar^2 k} B_1$

$\psi_1(-a) = \psi_2(a) = 0$

Fall „ungerade“: Mit $A_1 = A_2 = A$ haben wir den Ansatz

$\psi_1(x) = A \sin(kx) = \psi_2(x)$

$\psi_2(a) = A \sin(ka), \quad A \neq 0 \Rightarrow k = \frac{n\pi}{a}$

$\sqrt{\frac{2mE}{\hbar^2}} \quad \Rightarrow E_n = \frac{\pi^2 \hbar^2}{2ma^2} n^2$

$\int\limits_{-\infty}^{\infty} |\Psi(x)|^2 dx = 1 \quad \Rightarrow \int\limits_{-a}^{a} A^2 \sin^2(kx) dx = A^2 a = 1 \quad \Rightarrow A = \frac{1}{\sqrt{a}}$

Also für ungerade Parität: $\Psi_{ungerade}(x) = \sqrt{\frac{1}{a}} \sin(\frac{n\pi}{a} x)$

d) $\psi_1(-a) = \psi_2(a)$ für Fall „gerade“:

$-A_1 \sin(ka) + B_1 \cos(ka) = 0$

$A_2 \sin(ka) + B_2 \cos(ka) = 0$

$\Rightarrow B_1 \cos(ka) = A_1 \sin(ka) \quad \Rightarrow \frac{\sin(ka)}{\cos(ka)} = \tan(ka) = \frac{B_1}{A_1}$

$A_1 = A_2 = -\frac{mV_0}{\hbar^2 k} B_1 \quad \Rightarrow \boxed{\tan(ka) + \frac{\hbar^2 k}{mV_0} = 0}$

$\Rightarrow \;\; \tan(ka) + cka = 0$ mit $c = \frac{\hbar^2}{mV_0 a}$

e) $c << 1$:

$\Rightarrow -cx$ hat sehr geringe Steigung.

$\Rightarrow$ Schnitt von $\tan(x)$ und $-cx$ ungefähr bei π

$\tan(x) \approx (x-\pi) + O(x^3)$

$\Rightarrow x - \pi + cx = 0 \quad \Rightarrow x = \frac{\pi}{1+c} \quad \Rightarrow k \approx \frac{\pi}{a(1+c)}$

f) $\Rightarrow E = \frac{\hbar^2 k^2}{2m} \approx \frac{\hbar^2\pi^2}{2ma^2}\frac{1}{(1+c)^2}$

$\frac{1}{(1+c)^2} \approx 1 - 2c$ (kleine c)

$\Rightarrow E \approx \frac{\hbar^2\pi^2}{2ma^2}(1-2c)$

4.3 Zusatzaufgabe 2

$a = 1, c = 0.05$

$k = \frac{\pi}{1+0.05} \approx 2.991993...$

genau: 2.9930429

5 Tutorium vom 06.05.11 Klausur1-Besprechung

5.1 K2

2 Gebiete: $x < 0, x > 0$, jeweils $V(x) = 0$

$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi(x) \Rightarrow \frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi$

$\kappa := \sqrt{-\frac{2mE}{\hbar^2}}, \quad E < 0$

Lös.Ansatz (brücksichtigt Normierbarkeit):

$$\psi(x) = \begin{cases} Be^{\kappa x} & , < 0 \\ Ae^{-\kappa x} & x > 0 \end{cases}$$

Bei x=0:

i) Wellenfu' stetig $\Rightarrow B = A$

ii) Integration der Schrödingergl. um x=0:
$-\frac{\hbar^2}{2m}\int\limits_{-\varepsilon}^{\varepsilon}\frac{d^2\psi}{dx^2}dx + \int\limits_{-\varepsilon}^{\varepsilon} V_0\delta(x)\psi(x)dx = \int\limits_{-\varepsilon}^{\varepsilon} E\psi(x)dx \Rightarrow \frac{-\hbar^2}{2m}(\psi'(\varepsilon) - \psi'(-\varepsilon)) = -V_0\Psi(0) + E(\Psi(\varepsilon) - \Psi(-\varepsilon))\Psi$: Stammfu' von ψ

$\varepsilon \to 0$: Ableitung von $\psi(x)$ bei x=0 springt um $\frac{2mV_0}{\hbar^2}\psi(0)\}$

$$\text{Es ist } \psi'(x) = \begin{cases} \kappa Be^{\kappa x} & , x < 0 \\ -\kappa A^{-\kappa x} & x > 0 \end{cases}$$

Somit folgt $\psi'(0^+) - \psi(0^-) = -\kappa Ae^0 - \kappa Be^0$

$\Rightarrow -\kappa A - \kappa B = \frac{2mV_0}{\hbar^2}A$

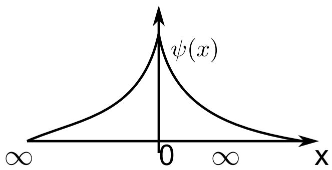
$\Rightarrow \kappa = -\frac{mV_0}{\hbar^2} = \sqrt{-\frac{2mE}{\hbar^2}} \Rightarrow E = -\frac{mV_0^2}{2\hbar}$

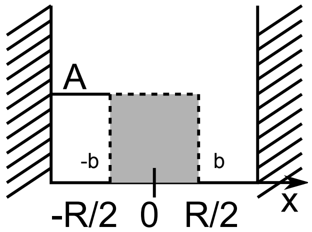
Normierung: $\int\limits_{-\infty}^{\infty} |\psi(x)|^2 dx = \int\limits_{-\infty}^{0} A^2 e^{2\kappa x} + \int\limits_{0}^{\infty} A^2 e^{-2\kappa x} dx$

$= A^2 \underbrace{[\frac{e^{\kappa x}}{2\kappa}]_{-\infty}^{0}}_{1/2\kappa} + A^2 \underbrace{[-\frac{e^{-2x\kappa}}{2\kappa}]_0^{\infty}}_{1/2\kappa} \stackrel{!}{=} 1$

$A = \sqrt{\kappa x}$

$$\psi(x) = \begin{cases} \sqrt{\kappa} e^{\kappa x} & , x < 0 \\ \sqrt{x} e^{-\kappa x} & , x > 0 \end{cases}$$





5.2 K3

$\psi(x,0) = A\theta(b^2 - x^2)$

a) Normierung:

$\int\limits_{-\frac{R}{2}}^{\frac{R}{2}} dx |\psi(x,0)|^2 \stackrel{!}{=} 1$

$\Rightarrow |A|^2 \cdot 2b = 1 \quad \Rightarrow A = \frac{1}{\sqrt{2b}}$

b) Spektrum des Potentialtopfes:

$\phi(\pm\frac{R}{2}) = 0 \quad \Rightarrow \phi(x) = A\cos(kx) + B\sin(kx)$

R.B.: $0 = \phi(-\frac{R}{2}) = A\cos(k\frac{R}{2}) - B\sin(k\frac{R}{2})$

$0 = \phi(\frac{R}{2}) = A\cos(k\frac{R}{2}) + B\sin(k\frac{R}{2})$

Das Potential ist achensymmetrisch um x=0

c) det $\begin{pmatrix} \cos(z) & -\sin(z) \\ \cos(z) & \sin(z) \end{pmatrix} = 0$ mit $z = \frac{kR}{2}$.

Also $D(z) = 2\sin(z)\cos(z). D(z_n) \stackrel{!}{=} 0$

$\Rightarrow \sin(z_n) = 0$ bzw. $z_n = n\pi \quad \Rightarrow k_n = \frac{2n\pi}{R}, \quad (n \in \mathbb{N} \geq 1)$

oder $\cos(z_n) = 0 \Rightarrow z_n = (2n+1)\frac{\pi}{2}, \quad k_n = (2n+1)\frac{\pi}{R}$

$\Rightarrow E_n = n^2 \frac{\hbar^2}{2m} (\frac{\pi}{R})^2 = n^2 E_1$ und $k_n = n\frac{\pi}{R} = nk_1$

Normierung: $\Rightarrow A_n = \sqrt{\frac{2}{R}}$

d) Also gute Parität:

gerade $\phi_n(x) = A_n \cos(k_n x), \quad \phi_n(-x) = +\phi_n(x)$

ungerade: $\phi_n(x) = A_n \sin(k_n x), \quad \phi_n(-x) = -\phi_n(x)$

e) $\{\varphi_n\}_{n=1,\ldots,\infty}$, VONS (vollständiges Orthonormales System)

$\sum_n |\varphi_n >< \varphi_n| = \mathbb{1}$

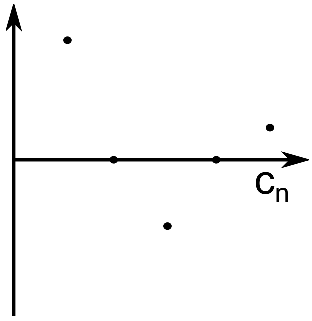
$\sum_n \varphi_n(x)\varphi^*(x') = \delta(x-x')$

$\psi(x,0) = \sum_n \varphi_n(x) \int_{-R/2}^{R/2} dx' \varphi_n^*(x')\psi(x',0)$

Also gilt: $c_n = \int_{-R/2}^{R/2} dx \varphi_n^*(x)\psi(x,0)$

$c_n = \frac{1}{\sqrt{2b}}\sqrt{\frac{2}{R}}\int_{-b}^{b} dx \begin{cases} \cos(k_n x) & P=1 \\ \sin(k_n x) & P=-1 \end{cases}$

$= \frac{1}{\sqrt{bR}}\frac{1}{k_n}\begin{cases} \sin(k_n x) \\ \cos(k_n x) \end{cases} \Big|_{-b}^{b} = \frac{1}{\sqrt{bR}}\frac{1}{k_n}\begin{cases} 2\sin(k_n b) \\ 0 \end{cases}$



f) $c_{2n} = 0$

$c_{2n+1} = \frac{1}{\sqrt{bR}}\frac{2R}{\pi}\frac{1}{2n+1}\sin((2n+1)\pi\frac{b}{R}) \underset{n\to\infty}{\longrightarrow} 0$

Wir finden also $\psi(x,0) = \sum_{n=0}^{\infty} c_{2n+1}\varphi_{2n+1}(x) = \frac{2}{\pi}\sqrt{\frac{R}{b}}\sum_{n\geq 0}\frac{1}{2n+1}\sin((2n+1)pi\frac{b}{R})\varphi_{2n+1}(x)$

g) Für beliebige Zeiten t gilt:

$\varphi_n(x) \mapsto \phi_n(x,t) = e^{-i\omega_n t}\varphi_n(x)$

und somit $\psi(x,t) = \sum_{n\geq 1} c_{2n+1}\phi_{2n+1}(x)$

$\psi(x,t) = \frac{2}{\pi}\sqrt{\frac{R}{b}}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin((2n+1)\pi\frac{b}{R})e^{-i\omega_{2n+1}t}\varphi_{2n+1}(x)$

5.3 Zusatz

$V(x) = \int_0^{\infty}\frac{p\sin(px)}{p^2+m^2}dp = \frac{1}{2}\int_{-\infty}^{\infty}\frac{p\sin(px)}{p^2+m^2}dp$

$e^{i\phi} = \cos(\phi)i\sin(\phi) \Rightarrow \sin(px) = \Im(e^{ipx})$

$V(x) = \frac{1}{2}\Im(\int_{e^{ipx}p}^{p^2+m^2} dp).$

Pole bei $p = \pm im \quad \Rightarrow V(x) = \frac{1}{2}\Im(2\pi i\cdot Res_f(im)) = \frac{1}{2}\Im(2\pi i\cdot\frac{ime^{-mx}}{2im}) = \frac{1}{2}\pi e^{-mx}$

6 Tutorium vom 03.06.11: Blatt 6

6.1 Hausaufgabe 12

a) $H = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2 - \gamma\sqrt{2m\hbar\omega^3}x$

$= " + \frac{m\omega^2}{2}(x^2 - 2\gamma\frac{\sqrt{2m\hbar\omega^3}x}{m\omega^2}$

$= " + \frac{m\omega^2}{2}(x^2 - 2\gamma\sqrt{\frac{2\hbar}{m\omega}}x)$

$= " + \frac{m\omega^2}{2}(x^2 - 2\gamma\sqrt{\frac{2\hbar}{m\omega}}x + \gamma^2\frac{2\hbar}{m\omega} - \gamma^2\frac{2\hbar}{m\omega})$

$= " + \frac{m\omega^2}{2}((x - \gamma\sqrt{\frac{2\hbar}{m\omega}})^2 - \gamma^2\frac{2\hbar}{m\omega})$

$$= \frac{p^2}{2m} + \frac{m\omega^2}{2}(x - \underbrace{\gamma\sqrt{\frac{2\hbar}{m\omega}}}_{=:x_0})^2 - \gamma^2\hbar\omega$$

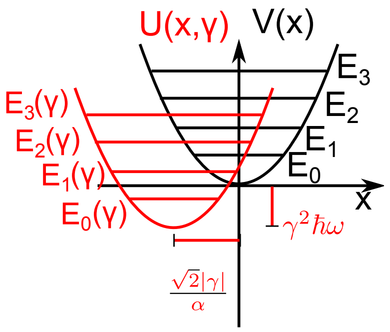
$x_0 = \gamma\sqrt{\frac{2\hbar}{m\omega}} = \gamma\frac{\sqrt{2}}{\alpha}$ mit $\alpha = \sqrt{\frac{m\omega}{\hbar}}$

$E_n(\gamma) = \hbar\omega(n + \frac{1}{2} - \gamma^2)$

$< x|\psi_n(\gamma) >= \sqrt{\frac{\alpha}{2^n n!\sqrt{\pi}}}\exp(-\frac{\alpha(x-x_0)^2}{2})H_n(\alpha(x - x_0))$

H_n: Hermite Polynome

b)



c) $H_1 = -\gamma\sqrt{2m\hbar\omega^3}x$

$= -\gamma\sqrt{2m\hbar\omega^3}\sqrt{\frac{\hbar}{2m\omega}}(b^\dagger + b) = -\gamma\hbar\omega(b^\dagger + b)$

$< n'|H_1|n >=< n'|-\gamma\hbar\omega(b^\dagger+b)|n >= -\gamma\hbar\omega < n'|(b^\dagger|n > +(-\gamma\hbar\omega) < n'|b|n >$

$= -\gamma\hbar\omega < n'|\sqrt{n+1}|n + 1 > -\gamma\hbar\omega < n'|\sqrt{n}|n - 1 >$

$= -\gamma\hbar\omega(\sqrt{n+1} < n'|n + 1 > +\sqrt{n} < n'|n - 1 >)$

$= -\gamma\hbar\omega(\sqrt{n+1}\delta_{n',n+1} + \sqrt{n}\delta_{n',n-1})$

7 Tutorium vom 10.06.11 (Blatt 7)

7.1 Hausaufgabe 13

a) $[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$

Alle aufführen:

$$[\sigma_1, \sigma_2] = \sigma_1\sigma_2 - \sigma_2\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} i & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} = 2i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2i\sigma_3$$

$[\sigma_2, \sigma_3] = 2i\sigma_1\varepsilon_{231} = 2i\sigma_1$

$[\sigma_1, \sigma_3] = 2i\sigma_2\varepsilon_{132} = -2i\sigma_2$

etc.

$$\{\sigma_i, \sigma_j\} = \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}\mathbb{1} = 2\delta_{ij}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\{\sigma_1,\sigma_2\} = \sigma_1\sigma_2 + \sigma_2\sigma_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\{\sigma_1,\sigma_1\} = 2\sigma_1\sigma_1 = 2\mathbb{1}$$

b) $[\sigma_i,\sigma_j] = 2i\varepsilon_{ijk}\sigma_k = \sigma_i\sigma_j - \sigma_j\sigma_i$

$\{\sigma_i,\sigma_j\} = 2\delta_{ij}\mathbb{1} = \sigma_i\sigma_j + \sigma_j\sigma_i$

„+“$\Rightarrow 2\sigma_i\sigma_j = 2(delta_{ij}\mathbb{1} + \varepsilon_{ijk}\sigma_k)$

$\sigma_i\sigma_j = \delta_{ij}\mathbb{1} + i\varepsilon_{ijk}\sigma_k$

$(\vec{a}\vec{\sigma})(\vec{b}\vec{\sigma}) = \vec{a}\vec{b}\mathbb{1} + i\sigma(\vec{a}\times\vec{b})$

$= a_i\sigma_i b_j\sigma_j = a_i b_j\sigma_i\sigma_j = a_i b_j(\delta_{ij}\mathbb{1} + i\varepsilon ijk\sigma_k) = a_i b_j\delta_{ij} + i\varepsilon_{ijk}a_i b_j\sigma_k = (\vec{a}\vec{b})\mathbb{1} +$
$i\vec{\sigma}(\vec{a}\times\vec{b})$

c) $e^{i\alpha\sigma_k} = \cos(\alpha) + i\sigma_k\sin(\alpha), \quad \alpha\in\mathbb{C}, \quad k=1,2,3$

$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_1^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1^3 = \mathbb{1}$

$e^{i\alpha\sigma_k} = \sum\limits_{n=0}^{\infty} \frac{i^n\alpha^n}{n!}\sigma_k^n$

$= \sum\limits_{n=0}^{\infty} \frac{(-1)^n\alpha^{2n}\sigma_k^{2n}}{(2n)!} + i\sum\limits_{n=0}^{\infty} \frac{(-1)^n(\alpha\sigma_k)^{2n+1}}{(2n+1)!}$

$= \mathbb{1}\sum\limits_{n=0}^{\infty} \frac{(-1)^n\alpha^{2n}}{(2n)!} + i\sigma_k\sum\limits_{n=0}^{\infty} \frac{(-1)^n\alpha^{2n+1}}{(2n+1)!}$

$= \mathbb{1}\cos(\alpha) + i\sigma_k\sin(\alpha)$

d) $M = m_0\mathbb{1} + \sum\limits_{i=1,2,3} m_i\sigma_i = \begin{pmatrix} m_0 & 0 \\ 0 & m_0 \end{pmatrix} + \begin{pmatrix} 0 & m_1 \\ m_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -im_2 \\ im_2 & 0 \end{pmatrix} +$
$\begin{pmatrix} m_3 & 0 \\ 0 & -m_3 \end{pmatrix} = \begin{pmatrix} m_0+m_3 & m_1-im_2 \\ m_1+im_2 & m_0-m_3 \end{pmatrix}$

$m_0 = \frac{tr(M)}{2}, \quad m_i = \frac{1}{2}tr(\sigma_i M)$

e) trivial

f) i) $\sigma_i^\dagger = (\sigma_i^*)^T$ hermitisch

$\sigma_2^\dagger = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}^T = \sigma_1$

$\sigma_3^\dagger = \sigma_1$

ii) $tr(\sigma_i) = 0$

iii) $\det(\sigma_i) = -1$

iv) $\sigma_i^T = (-1)^{i+1}\sigma_i$

7.2 Hausaufgabe 14

$\phi(x) = Axe^{-\frac{x^2}{2\gamma^2}}$

a) $\int\limits_{-\infty}^{\infty} dx|\phi(x)|^2 = 1 \ \Rightarrow \frac{1}{|A|^2} = \int\limits_{-\infty}^{\infty} dx x^2 e^{-\frac{x^2}{\gamma^2}} = -\frac{\partial}{\partial x}\Big|_{\frac{1}{\gamma^2}} \int\limits_{-\infty}^{\infty} dx e^{-\alpha x^2} = -\frac{\partial}{\partial \alpha}\Big|_{\frac{1}{\gamma^2}} \sqrt{\frac{\pi}{\alpha}} =$
$\big[\frac{1}{2}\frac{\sqrt{\pi}}{\alpha^{3/2}}\big]_{\alpha=\frac{1}{\gamma^2}}$
$= \gamma^3 \cdot \frac{1}{2}\sqrt{\pi} \quad \Rightarrow A = \sqrt{\frac{2}{\gamma^3\sqrt{\pi}}}$

b) $(H-E)\phi(x) = 0 = (T + V(x) - E)\phi(x)$

$\big(\frac{\partial^2}{\partial x^2} + \frac{2m}{\hbar^2}(E - V(x))\big)\phi(x) = 0$

$\phi' = A(1 - \frac{x^2}{\gamma^2})e^{-\frac{x^2}{2\gamma^2}}$

$\phi'' = A\big[-\frac{2x}{\gamma^2} - \frac{x}{\gamma^2}(1 - \frac{x^2}{\gamma^2})\big]e^{-\frac{x^2}{2\gamma^2}}$

$= A\big[-\frac{3}{\gamma^2}x + \frac{1}{\gamma^4}x^3\big]e^{-\frac{x^2}{2\gamma^2}}$

$= -\big[\frac{3}{\gamma^2} - \frac{x^2}{\gamma^4}\big]Axe^{-\frac{x^2}{2\gamma^2}}$

$\phi'' = -[\frac{3}{\gamma^2} - \frac{x^2}{\gamma^4}]\phi(x)$

d.h. $\big[\frac{\partial^2}{\partial x^2} + (\frac{3}{\gamma^2} - \frac{x^2}{\gamma^4})\big]\phi(x) = 0$

und damit $\frac{2m}{\hbar^2}(E - V(x)) = \frac{3}{\gamma^2} - \frac{x^2}{\gamma^4}$

Mit $V(0) = 0: \quad E = \frac{3}{2}\frac{\hbar^2}{m\gamma^2}, \quad V(x) = \frac{\hbar^2}{2m}\frac{x^2}{\gamma^4}$

Außerdem erkennt man: $\frac{1}{\gamma^2} = \frac{m\omega}{\hbar} \ \Rightarrow E = \frac{3}{2}\hbar\omega = [\hbar\omega(n + \frac{1}{2})]_{n=1}$

$\Rightarrow V(x) = \frac{1}{2}\hbar\omega\frac{x^2}{\gamma^2}$

$\Rightarrow \phi(x)$ ist der 1. angeregte Zustand in einem Oszillatorpotential

8 Tutorium vom 17.06.11 (Blatt 8)

P11

$$L_x = \begin{pmatrix} 0 & \hbar/\sqrt{2} & 0 \\ \hbar/\sqrt{2} & 0 & \hbar/\sqrt{2} \\ 0 & \hbar/\sqrt{2} & 0 \end{pmatrix}$$

a)

$$P_{L_x}(\lambda) = \det(L_x - \lambda \cdot \mathbb{1}_3) \quad \text{charakt. Polynom}$$

$$= \det \begin{pmatrix} -\lambda & \hbar/\sqrt{2} & 0 \\ \hbar/\sqrt{2} & -\lambda & \hbar/\sqrt{2} \\ 0 & \hbar/\sqrt{2} & -\lambda \end{pmatrix}$$

Entwicklung nach 1. Zeile

$$= -\lambda \begin{vmatrix} -\lambda & \hbar/\sqrt{2} \\ \hbar/\sqrt{2} & -\lambda \end{vmatrix} - \frac{\hbar}{\sqrt{2}} \begin{vmatrix} \hbar/\sqrt{2} & \hbar/\sqrt{2} \\ 0 & -\lambda \end{vmatrix}$$

$$= -\lambda \left(\lambda^2 - \frac{\hbar^2}{2}\right) - \frac{\hbar}{\sqrt{2}} \cdot \left(-\lambda \cdot \frac{\hbar}{\sqrt{2}}\right)$$

$$= -\lambda^3 + \lambda \frac{\hbar^2}{2} + \lambda \frac{\hbar^2}{2}$$

$$= -\lambda^3 + \lambda \hbar^2 = \lambda(-\lambda^2 + \hbar^2) \overset{!}{=} 0$$

$$\left.\begin{aligned} &\Rightarrow \lambda_1 = 0 \\ &\Rightarrow \lambda_2 = +\hbar \\ &\Rightarrow \lambda_3 = -\hbar \end{aligned}\right\} \text{Eigenwerte}$$

Bestimmung der Eigenvektoren:

b)

$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ ist Eigenvektor von L_x, wenn $\vec{x}$ nichttriviale

Lösung von $(L_x - \lambda \mathbb{1}_3)\vec{x} = \vec{0}$ ist, d.h. für $\lambda = 0$:

$$\begin{pmatrix} 0 & \hbar/\sqrt{2} & 0 \\ \hbar/\sqrt{2} & 0 & \hbar/\sqrt{2} \\ 0 & \hbar/\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Zeilen tauschen

Gauß-Algorithmus zum Lösen des LGS

$$\left(\begin{array}{ccc|c} \hbar/\sqrt{2} & 0 & \hbar/\sqrt{2} & 0 \\ 0 & \hbar/\sqrt{2} & 0 & 0 \\ 0 & \hbar/\sqrt{2} & 0 & 0 \end{array}\right) \quad \cdot(-1)\text{, } +$$

$$\left(\begin{array}{ccc|c} \hbar/\sqrt{2} & 0 & \hbar/\sqrt{2} & 0 \\ 0 & \hbar/\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \Rightarrow \frac{\hbar}{\sqrt{2}} x_1 = -\frac{\hbar}{\sqrt{2}} x_3 \Rightarrow \boxed{-x_1 = x_3} \\ \Rightarrow \frac{\hbar}{\sqrt{2}} \cdot x_2 = 0 \Rightarrow \boxed{x_2 = 0} \end{array}$$

Sei $x_3 = t$. Dann ist der Eigenvektor also von der Form $(-t, 0, t)$. Der normierte Eigenvektor lautet also $\left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ und gehört zum Eigenwert $\lambda_1 = 0$.

Analog erhält man durch das Lösen von LGS:

Eigenwert	normierter Eigenvektor
$-\hbar$	$\left(\frac{1}{2}, -\frac{1}{\sqrt{2}}, \frac{1}{2}\right) =: \vec{a}_1$
0	$\left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) =: \vec{a}_2$
$+\hbar$	$\left(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right) =: \vec{a}_3$

c)

Die Eigenvektoren bilden ein Orthonormalsystem:

$\vec{a} \cdot \vec{b} = 0 \Rightarrow \vec{a} \perp \vec{b}$

$$\vec{a}_1 \cdot \vec{a}_2 = \frac{1}{2} \cdot \left(-\frac{1}{\sqrt{2}}\right) + 0 + \frac{1}{2} \cdot \frac{1}{\sqrt{2}} = 0$$

$$\vec{a}_1 \cdot \vec{a}_3 = \frac{1}{2} \cdot \frac{1}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} - \frac{1}{2} + \frac{1}{4} = 0$$

$$\vec{a}_2 \cdot \vec{a}_3 = -\frac{1}{\sqrt{2}} \cdot \frac{1}{2} + 0 + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = 0$$

d) Bezüglich der Basis $\mathcal{B} = \{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ haben wir

$$L_x^{\mathcal{B}} = \begin{pmatrix} \vec{a}_1 \cdot L_x \cdot \vec{a}_1 & \vec{a}_1 \cdot L_x \cdot \vec{a}_2 & \vec{a}_1 \cdot L_x \cdot \vec{a}_3 \\ \vec{a}_2 \cdot L_x \cdot \vec{a}_1 & \vec{a}_2 \cdot L_x \cdot \vec{a}_2 & \vec{a}_2 \cdot L_x \cdot \vec{a}_3 \\ \vec{a}_3 \cdot L_x \cdot \vec{a}_1 & \vec{a}_3 \cdot L_x \cdot \vec{a}_2 & \vec{a}_3 \cdot L_x \cdot \vec{a}_3 \end{pmatrix}$$

$$L_x \cdot \vec{a}_1 = \begin{pmatrix} 0 & \hbar/\sqrt{2} & 0 \\ \hbar/\sqrt{2} & 0 & \hbar/\sqrt{2} \\ 0 & \hbar/\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{pmatrix} = \begin{pmatrix} -\hbar/2 \\ \hbar/\sqrt{2} \\ -\hbar/2 \end{pmatrix}$$

$$\vec{a}_1 \cdot (L_x \cdot \vec{a}_1) = \left(\frac{1}{2}, -\frac{1}{\sqrt{2}}, \frac{1}{2}\right) \cdot \left(-\frac{\hbar}{2}, \frac{\hbar}{\sqrt{2}}, -\frac{\hbar}{2}\right)$$

$$= -\frac{\hbar}{4} - \frac{\hbar}{2} - \frac{\hbar}{4} = -\hbar = \left(L_x^{\mathcal{B}}\right)_{11}$$

Durch 8 analoge Rechnungen findet man schließlich

$$L_x^{\mathcal{B}} = \begin{pmatrix} -\hbar & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \hbar \end{pmatrix}$$

H15

$x, y, z \in \mathbb{R}^+$,

$x + y + z \leq 3$.

Sei $x_1 = \sqrt{x}$, $y_1 = \sqrt{\frac{1}{x}}$,

$x_2 = \sqrt{y}$, $y_2 = \sqrt{\frac{1}{y}}$,

$x_3 = \sqrt{z}$, $y_3 = \sqrt{\frac{1}{z}}$.

Cauchy-Schwarz: $(x_1^2 + x_2^2 + x_3^2)(y_1^2 + y_2^2 + y_3^2) \geq (x_1y_1 + x_2y_2 + x_3y_3)^2$

für reelle x_i und y_j

Es ist $(x_1y_1 + x_2y_2 + x_3y_3)^2 = (1+1+1)^2 = 3^2 = 9$.

Also folgt $9 \leq (x+y+z) \cdot \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)$

Teile diese Ungleichung durch $x+y+z$ (positive Zahl $\neq 0$).

$\frac{9}{x+y+z}$ ist größer oder gleich 3, weil der Nenner höchstens 3 ist.

Also: $3 \leq \frac{9}{x+y+z} \leq \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$,

was zu beweisen war.

H16) a), b), c): viele Matrixrechnungen, siehe extra pdf mit Mathematica-Rechnungen

d), e), f): analog Präsenzaufgabe, siehe extra pdf.

<u>Zusatzaufgabe</u>

Es ist $a = \sqrt{\frac{m\omega}{2\hbar}}\underbrace{x}_{Q=\hat{x}} + i \cdot \frac{1}{\sqrt{2\hbar m\omega}} \cdot \underbrace{\frac{\hbar}{i}\frac{\partial}{\partial x}}_{P=\hat{p}}$

$a = \sqrt{\frac{m\omega}{2\hbar}} Q + i \frac{P}{\sqrt{2\hbar m\omega}}$; $a^{\dagger} = \sqrt{\frac{m\omega}{2\hbar}} Q - i \frac{P}{\sqrt{2\hbar m\omega}}$

Paritätsoperator $\mathcal{P}$:

$$\mathcal{P} a = \sqrt{\frac{m\omega}{2\hbar}}(-x) + i \cdot \frac{1}{\sqrt{2\hbar m\omega}} \frac{\hbar}{i}\left(-\frac{\partial}{\partial x}\right)$$

$$= -\left(\sqrt{\frac{m\omega}{2\hbar}} x + i \frac{1}{\sqrt{2\hbar m\omega}} \frac{\hbar}{i} \frac{\partial}{\partial x}\right) = -a$$

$\Rightarrow$ negative Parität

$$\mathcal{P} a^{\dagger} = \sqrt{\frac{m\omega}{2\hbar}}(-x) - i \frac{1}{\sqrt{2\hbar m\omega}} \frac{\hbar}{i}\left(-\frac{\partial}{\partial x}\right)$$

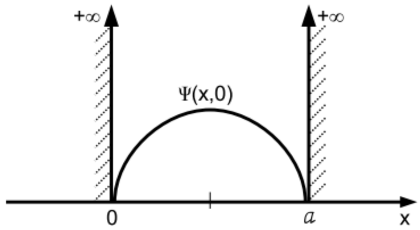
$$= -\sqrt{\frac{m\omega}{2\hbar}} x + i \frac{1}{\sqrt{2\hbar m\omega}} \frac{\hbar}{i} \frac{\partial}{\partial x}$$

$$= -\left(\sqrt{\frac{m\omega}{2\hbar}} x - i \frac{1}{\sqrt{2\hbar m\omega}} \frac{\hbar}{i} \frac{\partial}{\partial x}\right) = -a^{\dagger} \Rightarrow \text{negative Parität}$$

Aufgrund der negativen Parität kann es keine von 0 verschiedenen Diagonalmatrixelemente geben.

9 Tutorium vom 24.06.11 (Blatt 9)

9.1 Hausaufgabe 17



$\psi(x,0) = Ax(a-x)$

a) $1 = A^2 \int_0^a dx x^2 (a-x)^2 = A^2 \int_0^a dx x^2 (a^2 - 2ax + x^2)$

$= A^2 \int_0^a dx (a^2x^2 - 2ax^3 + x^4) = A^2 (\frac{a^5}{3} - \frac{2}{4}a^5 + \frac{1}{5}a^5)$

$= A^2 a^5 (\frac{1}{3} - \frac{1}{2} + \frac{1}{5}) = A^2 a^5 \frac{10-15+6}{30} = A^2 \frac{a^5}{30}$

$\Rightarrow A = \sqrt{\frac{30}{a^5}} = \frac{1}{a^2}\sqrt{\frac{30}{a}}$

$\psi(x,0) = (\pm)\sqrt{\frac{30}{a^5}} x(a-x)$

b) Entweder nach VON des Topfes:

$\phi_n(x) = \sqrt{\frac{2}{a}} \sin(\frac{n\pi}{a}x), \ \ n = 1, 2..., \ \ \ 0 \leq x \leq a$

$\psi(x,0) = \sum_n c_n \phi_n(x)$

$c_n = \sqrt{\frac{2}{a}} A \int_0^a dx x(a-x) \sin(k_n x), \ \ k_n = \frac{n\pi}{a}$

$= A\sqrt{\frac{2}{a}} \int_0^a dx (ax - x^2) \sin(k_n x)$

1) $\int_0^a dx x \sin(k_n x) = -\frac{1}{k_n} \int_0^a dx x \frac{d}{dx} \cos(k_n x)$

$= -\frac{1}{k_n} a \cos(k_n a) + \frac{1}{k_n} \int_0^a dx \cos(k_n x) = -\frac{1}{k_n} a \cos(k_n a) + \frac{1}{k_n^2} \underbrace{\sin(k_n a)}_{=0}$

$= -\frac{1}{k_n} a \cos(n\pi) = -(-1)^n \frac{a^2}{n\pi}$

2) $\int_0^a dx x^2 \sin(k_n x) = -\frac{\partial^2}{\partial k_n^2} \int_0^a dx \sin(k_n x)$

$\overset{s=k_n a = n\pi}{=} \frac{\partial^2}{\partial k_n^2} \frac{1}{k_n} (\cos(k_n a) - 1) = a^3 \frac{d^2}{ds^2} (\frac{\cos(s)-1}{s})$

$\frac{d^2}{dx^2}(g(x)h(x)) = g''h + 2h'g' + hg''$

$\Rightarrow \frac{d^2}{ds^2} \frac{\cos(s)-1}{s} = \frac{2}{s^3}(\cos(s) - 1) + \frac{2}{s^2}\sin(s) - \frac{1}{s}$

$\frac{2}{(n\pi)^3}((-1)^n - 1) - \frac{1}{n\pi}(-1)^n = \int_0^a dx x^2 \sin(k_n x) \frac{1}{a^3}$

$$= \begin{cases} -\frac{1}{n\pi} & , n = 2k, \ \ k \in \mathbb{N} \\ -\frac{4}{(n\pi)^3} + \frac{1}{n\pi} & , n = 2k+1, \ \ k \in \mathbb{N} \end{cases}$$

Damit: $c_n = \sqrt{\frac{2}{a}} A a^3 \left[(-1)^{n+1}(\frac{1}{n\pi} - \frac{1}{n\pi}) + \frac{4}{(n\pi)^3}\frac{1}{2}(1 - (-1)^n)\right]$

$= \sqrt{\frac{2}{a}} a^3 A \frac{4}{n^3\pi^3} \frac{1}{2}(1 - (-1)^n)$

$c_{2k+1} = \sqrt{60} \frac{4}{\pi^3} \frac{1}{(2k+1)^3}, \ \ \ c_{2k} = 0$

$\psi(x,0) = \sqrt{15} \frac{8}{\pi^3} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} \phi_{2k+1}(x)$

c) Berechnung der Wahrscheinlichkeiten:

$\omega_1 = |c_1|^2 = \frac{15\cdot 64}{\pi^6} = \frac{16\cdot 60}{\pi^6} = \frac{960}{\pi^6} \approx 0.9986$

$\omega_1 = 1 - \bar{\omega} = 1.44 \cdot 10{-}3$

Norm der Reihe:

$\int_0^a dx|\psi(x,0)|^2 = \frac{960}{\pi^6}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^6} = \frac{(2^6-1)\pi^6}{6!\cdot 2}\frac{960}{\pi^6}B_6 = \frac{63\pi^6}{720\cdot 2}\frac{1}{42}\frac{960}{\pi^6} = \frac{63\cdot 60}{45\cdot 2\cdot 42} = 1$

d) Zeitentwicklung:

$\psi(x,t) = \frac{\sqrt{960}}{\pi^3}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}\phi_{2k+1}e^{-i\omega_{2k+1}t}$

$\omega_n = \frac{1}{\hbar}E_n = \frac{\hbar}{2m}\frac{\pi^2}{a^2}n^2$

e) Wahrscheinlichkeitsdichte, -stromdichte:

$\rho(x,t) = \sum_{n,k} c^*_{2k+1}\varphi^*_{2k+1}(x)c_{2n+1}\varphi_{2n+1}e^{i(\omega_{2k+1}-\omega_{2n+1})t}$

$c_j, \varphi_j(x) \in \mathbb{R}$

$\rho(x,t) = \rho_s(x) + \rho_t(x,t)$

$\rho_s(x) = \sum_n c^2_{2n+1}\varphi^2_{2n+1}(x)$

$\rho_t(x,t) = \sum_{k<n} c_{2k+1}c_{2n+1}\varphi_{2k+1}(x)\varphi_{2n+1}(x)(e^{i(\omega_{2k+1}-\omega_{2n+1})t} + e^{-i(\omega_{2k+1}-\omega_{2n+1})t})$

$= 2\sum_{k<n} c_{2k+1}c_{2n+1}\varphi_{2k+1}(x)\varphi_{2n+1}(x)\cos((\omega_{2k+1}-\omega_{2n+1})t)$

$j(x,t) = \frac{\hbar}{2im}\sum_{n,k}(\varphi^*_{2k+1}(x)\frac{d}{dx}\varphi_{2n+1}(x) - \varphi_{2n+1}\frac{d}{dx}\varphi^*_{2k+1}(x))c_{2k+1}c_{2n+1}e^{i(\omega_{2k+1}-\omega_{2n+1})t}$

$= \frac{\hbar}{2im}\sum_{k<n}(\varphi_{2n+1}(x)\frac{d}{dx}\varphi_{2n+1}(x) - \varphi_{2n+1}(x)\frac{d}{dx}\varphi_{2k+1}(x))c_{2k+1}c_{2n+1}(e^{i(\omega_{2k+1}-\omega_{2n+1})t} - e^{-i(\omega_{2k+1}-\omega_{2n+1})t})$

$= \frac{\hbar}{m}\sum_{k<n}(\varphi_{2n+1}(x)\frac{d}{dx}\varphi_{2n+1}(x) - \varphi_{2n+1}(x)\frac{d}{dx}\varphi_{2k+1}(x))c_{2k+1}c_{2n+1}\sin((\omega_{2k+1} - \omega_{2n+1})t)$

$\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}j(x,t) = 0$ Kontinuitätsgl. ist erfüllt:

$\frac{\partial}{\partial t}\rho(x,t) = \frac{\partial}{\partial t}\rho_t(x,t)$

$= -\frac{2}{\hbar}\sum_{k<n}(E_{2k+1} - E_{2n+1})c_{2k+1}c_{2n+1}\varphi_{2k+1}(x)\varphi_{2n+1}(x)\sin((\omega_{2k+1}-\omega_{2n+1})t)$

und mit $\varphi''_j(x) = -q_j^2\varphi_j(x)$ ergibt sich:

$\frac{\partial}{\partial x}j(x,t) = \frac{\hbar}{m}\sum_{k<n}(q^2_{2k+1} - q^2_{2n+1})\varphi_{2k+1}(x)\varphi_{2n+1}(x)c_{2k+1}c_{2n+1}\sin((\omega_{2k+1}-\omega_{2n+1})t)$

Also:

$\frac{\partial}{\partial x}j(x,t) = \frac{2}{\hbar}\sum_{k<n}(E_{2k+1} - E_{2n+1})\varphi_{2k+1}(x)\varphi_{2n+1}(x)c_{2k+1}c_{2n+1}\sin((\omega_{2k+1}-\omega_{2n+1})t)$

f) Berechnung von $< x > (t)$:

$< x > (t) = \int_0^a dx\psi^*(x,t)x\psi(x,t)$

$x = (x - \frac{a}{2}) + \frac{a}{2}$

$< x > (t) = \frac{960}{\pi^6}\sum_{k,k'}\frac{1}{(2k+1)^3(2k'+1)^3}\left[\frac{a}{2}\delta_{kk'} + \int_0^a dx(x-\frac{a}{2})\phi_{2k+1}(x)\phi_{2k'+1}(x)e^{i(\omega_{2k'+1}-\omega_{2k+1})t}\right]$

Da nur Funktionen auftreten, die Symmetrisch um $x = \frac{a}{2}$ sind, verschwindet das 2. Integral und $< x > (t)$ ist stationär.

$< x > (t) = \frac{960}{\pi^6}\frac{a}{2}\sum\limits_k \frac{1}{(2k+1)^6} = \frac{a}{2}$

Schwankung der Energie:

$(\Delta E)^2 =< (H- < H >)^2) >=< (H^2- < H >^2) >$

$< H >= \int\limits_0^a dx\psi^*(x,t)H\psi(x,t) = -i\hbar\int\limits_0^a dx\psi^*(x,t)\frac{\partial}{\partial t}\psi(x,t)$

$= \frac{960}{\pi^6}\sum\limits_k \frac{1}{(2k+1)^6}E_{2k+1} = \frac{960}{\pi^6}\frac{\hbar^2}{2m}\frac{\pi^2}{a^2}\sum\limits_k \frac{1}{(2k+1)^4} = E_1\frac{960}{\pi^6}\frac{(2^4-1)\pi}{4!2}|B_4|$

$= E_1\frac{960}{\pi^2}\frac{15}{48\cdot 30} = \frac{10}{\pi^2}E_1 \approx 1.01E_1$

Zu $< H^2 >$: $H^2\phi_n = H(H\phi_n) = E_nH\phi_n = E_n^2\phi_n$

$< H^2 >= \frac{960}{\pi^6}E_1^2\sum\limits_{k=0}^{\infty}\frac{(2k+1)^4}{(2k+1)^6} = \frac{960}{\pi^6}E_1^2\sum\limits_{k=0}^{\infty}\frac{1}{(2k+1)^2}$

$= \frac{960}{\pi^6}E_1^2\frac{(2^2-1)\pi^2}{2!\cdot 2}|B_2|E_1^2\frac{960}{\pi^4}\frac{3}{4\cdot 6} \approx 1.23E_1^2$

$\Delta E = \sqrt{< H^2 > - < H >^2} = E_1\sqrt{\frac{120}{\pi^4} - \frac{100}{\pi^4}} = E_1\frac{\sqrt{20}}{\pi^2}$

$\Delta E = 2E_1\frac{\sqrt{5}}{\pi^2} \approx 0.453E_1$

10 Tutorium vom 01.07.11

10.1 Z5

a) $G = \frac{1}{E+i\eta-H} \quad \eta \to 0+$

$H = \frac{\vec{p}^2}{2m} = -\frac{\hbar^2}{2m}\vec{\nabla}^2 = -\frac{\hbar^2}{2m}\frac{d^2}{dr^2}$

$H\psi = e\psi$

$-\hbar^2/2m\frac{d^2}{dr^2}\psi = e\psi$

$\psi'' + \frac{2me}{\hbar^2}\psi = 0$

$k^2 = \frac{2me}{\hbar^2} \quad \Rightarrow e = \frac{\hbar^2k^2}{2m} = e(k)$

$\psi(\vec{r}) = e^{i\vec{k}\vec{r}}$

$< \vec{k}|\vec{k}' >= \int \delta^3(\vec{k}-\vec{k}')d^3k$

$< \vec{k}|\vec{k} >= 1$

$< \vec{r}|\vec{k} >= \frac{1}{(2\pi)^{3/2}}e^{i\vec{k}\vec{r}}$

$=< \vec{r}|\frac{1}{E+i\eta-H}|\vec{r}' >= \int d\vec{k} < \vec{r}|\vec{k} > \frac{1}{E+i\eta-e} < \vec{k}|\vec{r}' > = \int \frac{1}{(2\pi)^3}d^3ke^{i\vec{k}\vec{r}}\frac{1}{E+i\eta-e(k)}e^{-i\vec{k}\vec{r}'} =$

$\int \frac{1}{(2\pi)^3}d^3ke^{i\vec{k}(\vec{r}-\vec{r}')}\frac{1}{E+i\eta-e(k)}$

b) $\int \frac{1}{(2\pi)^3}d^3ke^{i\vec{k}(\vec{r}-\vec{r}')}\frac{1}{E+i\eta-e(k)}r - \vec{r}' = \vec{s}, \quad \vec{k} = k\vec{e}_z$

sphärische Koord: ... $= \frac{1}{(2\pi)^3}\int\limits_0^\infty\int\limits_0^{2\pi}\int\limits_{-1}^{1} dkk^2d\cos(\theta)d\varphi e^{iks\cos(\theta)}\frac{1}{E+i\eta-e(k)}$

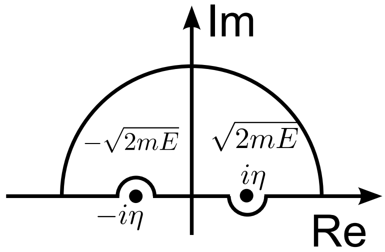
$= \frac{1}{4\pi^3}\int\limits_{-\infty}^{0} dkk^2\frac{e^{iks}-e^{-iks}}{iks}\frac{1}{E+i\eta-e(k)}$

$$= \frac{1}{4\pi^2}\int_{-\infty}^{\infty} dk \frac{ke^{iks}}{is}\frac{1}{E+i\eta-e(k)} = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} dk \frac{ke^{iks}}{is}\frac{1}{E+ieta-\frac{\hbar^2}{2m}k^2}$$

$$= \frac{m}{2\pi^2\hbar^2}\int_{-\infty}^{\infty} dk \frac{ke^{iks}}{is}\frac{1}{\frac{2m}{\hbar^2}E+i\cdot 0-k^2}$$

$$= \left| \begin{array}{ll} E>0,\ dann & \frac{2mE}{\hbar^2}-k^2=0 \\ & \hbar k_{1,2} = \pm\sqrt{2mE+i\eta} \\ (\eta\to 0_+) & \hbar k_1 = \sqrt{2mE}+i\eta \\ & \hbar k_2 = -\sqrt{2mE}+i\eta \end{array} \right.$$

c) $\hbar k_1 = \sqrt{2mE}+i\eta$ lies inside the contour



$$\Rightarrow \frac{m}{2\hbar\pi^2}\int_{-\infty}^{\infty} dk \frac{k}{\frac{2mE}{\hbar^2}-k^2}\frac{e^{iks}}{is}$$

$$= \frac{m}{2\hbar^2\pi^2}\cdot 2\pi i Res_{\frac{\sqrt{2mE}}{\hbar}}\left(\frac{k}{\frac{2mE}{\hbar^2}-k^2} - \frac{e^{iks}}{is}\right)$$

$$= -\frac{m}{2\pi\hbar}\frac{e^{is\frac{\sqrt{2mE}}{\hbar}}}{s}$$

$$G(\vec{r},\vec{r}') = -\frac{m}{2\pi\hbar^2}\exp\left(\frac{i\frac{\sqrt{2mE}}{\hbar}(\vec{r}-\vec{r}')}{(\vec{r}-\vec{r}')}\right)$$

$$G(s) \sim -\frac{e^{is}}{s} \sim \frac{\cos(s)}{s} + i\frac{\sin(s)}{s}$$

which exhibitsthe outgoing wave behavior from the sourcepoint: $\vec{r}' \to \vec{r}$

10.2 Minitest 6

a) $\{|a_1>, |a_2>\}, \quad <a_i|a_j> = \delta_{ij}$

$P_i = |a_i><a_i|$

$P_i^2|a_i> \underbrace{<a_i|a_i>}_{=1} <a_i| = |a_i><a_i| = P_i$

b) $p_i^\dagger = (|a_i><a_i|)^\dagger = P_i$

c) def.: $|a_i> \to \vec{e}_i = \begin{pmatrix} <a_1|e> \\ <a_2|e> \end{pmatrix}$

Im R^2: $(2\times 2)-Matrix$

$<a_i| \to (|a_i>)^\dagger \to |<a_i|e>^* <a_2|e>^*$

$a_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad a_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$P_1 = |a_1><a_1| = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

10.3 Hausaufgabe 18

a) $x\phi_s(x) = x\delta(x-s) = s\phi, \quad$ also EW: $x = s$

b) Die stationäre Impulseigenfu' sind bekannt: ebene Wellen

$\phi_k(x) = e^{ikx}$. Und es gilt bekanntlich

$\phi_s(x) = \delta(x-s) = \int\limits_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-s)} = \int\limits_{-\infty}^{\infty} \frac{dk}{2\pi} c_k(s) e^{iks}$

Also EW-Koeff $c_k(s) = e^{-iks}$

Kerim's Lösung:

a) $x\delta(x-s) = \lambda\delta(x-s) \quad | \int\limits_{-\infty}^{\infty} dx$

$$\underbrace{\int\limits_{-\infty}^{\infty} x\delta(x-s)dx}_{s} = \underbrace{\int\limits_{-\infty}^{\infty} \lambda\delta(x-s)dx}_{=\lambda}$$

$\Rightarrow EW: \quad \lambda = s$

b) $\hat{o}_x\phi(x) = p\phi(x)$

$-i\hbar\frac{\partial}{\partial x}\phi(x) - p\phi(x) = 0 \Leftrightarrow \phi'(x) + \frac{p}{i\hbar}\phi(x) = 0$

$\Rightarrow \phi(x) = \phi_p(x) = e^{i\frac{p}{\hbar}x}, \quad p \subset \mathbb{R}$

$\phi_s(x) = \delta(x-s) = \int\limits_{-\infty}^{\infty} \hat{\phi}_s(k)e^{iks}dk$

FT: $\hat{\phi}_s(k) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \phi_s(x)e^{-ikx}dx$

$= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \delta(x-s)e^{-ikx} = \frac{1}{2\pi}e^{-ikx}$

$\phi_s(x) = \int\limits_{-\infty}^{\infty} \frac{1}{2\pi}e^{-iks}e^{ikx}dk \underset{k=\frac{p}{\hbar}}{=} \int\limits_{-\infty}^{\infty} \frac{1}{2\pi\hbar}e^{-i\frac{p}{\hbar}s}\underbrace{e^{i\frac{p}{\hbar}x}}_{\phi_p(x)} dp$

Der Entwicklungskoeffizient ist somit $\frac{1}{2\pi\hbar}e^{-i\frac{p}{\hbar}s}$

11 Tutorium vom 08.07.2011 (2. Klausur)

11.1 K2

a) $V(r) = -\frac{Ze^2}{r}$

Schrödinger: $(T + V(r) - E_0)\psi(\vec{r}) = 0$

Elektron(en)sin vernachlässigt, Elektronen im 1s-Orbital:

$\psi(\vec{r}) = \frac{\gamma^{\frac{3}{2}}}{\sqrt{\pi}}e^{-\gamma r}$

$\gamma = \frac{Zme^2}{\hbar^2} = Z\frac{e^2}{\hbar c}\frac{mc^2}{\hbar c} \; \Big(= Z\alpha_f\frac{mc^2}{\hbar c}$

$\alpha_f = \frac{e^2}{\hbar c} \approx \frac{1}{137{,}03}\Big)$

Sei nun $Z \to Z+1 : \tilde{V}(r) = -\frac{(Z+1)e^2}{r} = V(r) - \frac{e^2}{r}$

$T + V(r) - \frac{e^2}{r} - E_0']\tilde{\phi}_0(\vec{r}) = 0$

$V_{eff} = V(r) + \frac{l(l+1)}{r^2}\frac{\hbar^2}{2m}$

b) $\Delta E^{(1)} =< \phi_0| - \frac{e^2}{r}|\phi_0 >$

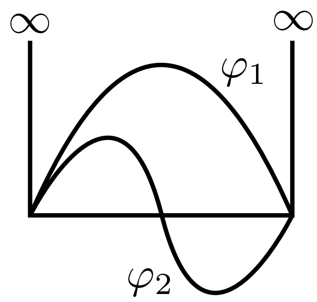
c) $\Delta E^{(1)} = -\frac{\gamma^3}{\pi}e^2 \int\limits_0^\infty drr^2\frac{1}{r}e^{-2\gamma r} \underbrace{\int d\Omega_r}_{4\pi}$

$= -e^2 4\gamma^3 \int\limits_0^\infty drre^{-2\gamma r} = -e^2 4\gamma^3(-)\frac{\partial}{\partial\gamma}|_{2\gamma} \int\limits_0^\infty dre^{-\beta r} = -e^2 4\gamma^3(-)\frac{\partial}{\partial\beta}|_{2\gamma}\frac{1}{\beta} = e^2 4\gamma^3\frac{1}{4\gamma^2} =$
$-\gamma e^2 = -\frac{Zme^4}{\hbar^2}$

$\Rightarrow \Delta E^{(1)} = -\alpha_f \hbar c\gamma = -Z\alpha_f^2 mc^2$

$\Rightarrow$ also stärkere Attraktion

11.2 K3



2-Zustands-Wellenpaket im unendlich tiefen Potential

Es gilt: $\psi(x,0) = A\sqrt{a}(\varphi_1(x) + z\varphi_2(x)), \quad z \in \mathbb{R}$

mit $\int\limits_{-a}^{a} dx|\varphi_i(x)|^2 = 1,$

$\varphi_1 = \frac{1}{\sqrt{a}}\cos(\frac{\pi}{2}\frac{x}{a}) = \varphi_1(-x), \quad \varphi_2 = \frac{1}{a}\sin(\pi\frac{x}{a}) = -\varphi_2(-x)$

a) $< \psi|\psi >= A^2 a(1+z^2) \stackrel{!}{=} 1 \quad \Rightarrow A = \frac{1}{\sqrt{a}}\frac{1}{\sqrt{1+z^2}}$

Also $\psi(x,0) = \frac{1}{\sqrt{1+z^2}}(\varphi_1(x) + z\varphi_2(x))$

b) Parität: $\psi(-x,0) = \frac{1}{\sqrt{1+z^2}}(\varphi_1(-x)z\varphi_2(-x)) = \frac{1}{\sqrt{1+z^2}}(\varphi_1(x) - z\varphi_2(x))$

$\Rightarrow$ keine gute Parität!

c) $\psi(x,0) \stackrel{\varphi_n\ VONS}{=} \sum\limits_n c_n\varphi_n = \sum\limits_n < \varphi_n|\psi > \varphi_n, \quad \omega_n = |c_n|^2 \omega_1 = \frac{1}{1+z^2}, \quad \omega_2 =$
$\frac{z^2}{1+z^2} = 1 - \omega_1, \quad \Rightarrow \omega_{n>1} = 0$

$c_n =< \varphi_n|\psi >= \int\limits_{-a}^{a} \varphi_n(x)\psi(x)dx$

d) $\psi(x,t) = \frac{1}{\sqrt{1+z^2}}(\varphi_1(x)e^{-i\omega_1 t} + z\varphi_2(x)e^{-i\omega_2 t})$

$\omega_1 = \frac{\hbar}{2m}(\frac{\pi}{2a})^2, \quad \omega_2 = \frac{\hbar}{2m}(\frac{\pi}{a})^2$

e) $< x >=< \psi|x|\psi >= \int\limits_{-a}^{a} dx\rho(x,t)x$

$(\Delta E)^2 =< H^2 > - < H >^2 =< \psi|H^2|\psi > - < \psi|H|\psi >^2$

$< \psi|H|\psi >= \frac{1}{1+z^2} < \varphi_1 + z\varphi_2|H|\varphi_1 + z\varphi_2 >$

$= \frac{1}{1+z^2}(< \varphi_1|H|\varphi_1 > + z^2 < \varphi_2|H|\varphi_2 >) = \frac{1}{1+z^2}(E_1 + z^2 E_2)$

$(\Delta E)^2 =< H^2 > - < H >^2 = \frac{1}{1+z^2}(E_1^2 + z^2 E_2^2 - (E_1 + z^2 E_2)^2\frac{1}{1+z^2})$

$= \frac{1}{(1+z^2)^2}(E_1^2(1+z^2-1) + E_2^2(z^2(1+z^2) - z^4) - 2z^2 E_1 E_2)$

$E_1 = \frac{\pi^2\hbar^2}{8ma^2}, \quad E_2 = \frac{\pi^2\hbar^2}{2ma^2}$

$... = (\frac{z^2}{1+z^2})^2(E_1^2 + E_2^2 - 2E_1E_2) = (\frac{z^2}{1+z^2})^2(E_1 - E_2)^2$

$= (\frac{z^2}{1+z^2})^2(\frac{\hbar^2}{2m}(\frac{\pi}{a})^2)^2((\frac{1}{2})^2 - 1)^2 = (\frac{z^2}{1+z^2})^2\frac{9}{16}(\frac{\hbar^2}{2m}(\frac{\pi}{a})^2)^2$

p) Zu $< p^2 >$:

$\frac{p^2}{2m} + V(x) = H$

$T + V = H$

$< p^2 >= 2m < T > \overset{V(x)=0}{=} 2m < H >$

oder händisch: $< \psi|(\frac{\hbar}{i}\frac{\partial}{\partial x})^2|\psi >$