Numerik 2 Blatt 3 Julian Bergmann

MSc Physik UB 10^{00}

Aufgabe 1

$$y''' - x^2 y'' + 4y' - xy = 3$$

$$z_0 = y, \quad z_1 = y', \quad z_2 = y''$$

$$\Rightarrow z_0' = z_1 = y', \quad z_1' = y'' = z_2$$

$$z_2' = y''' = 3 + x^2 y'' - 4y' + xy = 3 + x^2 z_2 - 4z_1 + x z_0$$

$$\Rightarrow \vec{z}\,' = \begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix}' = \begin{pmatrix} y' \\ y'' \\ y''' \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ 3 + x^2 z_2 - 4z_1 + x z_0 \end{pmatrix} = \vec{f}(x, \vec{z})$$

$$\vec{z}(1) = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$\vec{z}_{k+1} = \vec{z}_k + \frac{1}{2}\left(\vec{f}(x_k, \vec{z}_k) + \vec{f}(x_{k+1}, \vec{z}_{k+1}) \right)$$

$$\vec{z}_2 = \vec{z}_1 + \frac{1}{2}\left(\vec{f}\left(1, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}\right) + \vec{f}(2, \vec{z}_2) \right)$$

$$= \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \frac{1}{2}\left(\begin{pmatrix} 1 \\ 2 \\ 3 + 1^2 \cdot 2 - 4 \cdot 1 + 1 \cdot 1 \end{pmatrix} + \begin{pmatrix} z_1 \\ z_2 \\ 3 + 2^2 z_2 - 4z_1 + 2 z_0 \end{pmatrix} \right)$$

$$= \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 + z_1 \\ 2 + z_2 \\ 2 + 3 + 4z_2 - 4z_1 + 2z_0 \end{pmatrix}$$

$$\Rightarrow \left| \begin{array}{l} z_0 = 1{,}5 + 0{,}5 z_1 \\ z_1 = 2 + 0{,}5 z_2 \\ z_2 = 4{,}5 + 2 z_2 - 2 z_1 + z_0 \end{array} \right|$$

$$\Leftrightarrow \left| \begin{array}{l} z_0 = 2{,}5 + 0{,}25 z_2 \\ z_1 = 2 + 0{,}5 z_2 \\ z_2 = 4{,}5 + 2 z_2 - 2(2{,}5 + 0{,}25 z_2) + 2 + 0{,}5 z_2 \end{array} \right|$$

$$\Leftrightarrow \left| \begin{array}{l} z_0 = -0{,}5 \\ z_1 = -4 \\ z_2 = -12 \end{array} \right| \quad \Rightarrow \quad \vec{z}_2 = \begin{pmatrix} -0{,}5 \\ -4 \\ -12 \end{pmatrix} \approx \begin{pmatrix} y(2) \\ y'(2) \\ y''(2) \end{pmatrix}$$

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Aufgabe 2

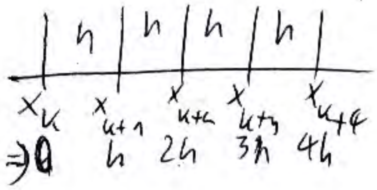
$$y(x_{k+4}) = y(x_{k+3}) + \int_{x_{k+3}}^{x_{k+4}} f(x, y(x))\, dx$$

$$P_3(x) = \sum_{j=0}^{3} f(x_{k+j}, y_{k+j}) L_{k+j}(x)$$

$$L_i(z) = \prod_{\substack{l=0 \\ l \neq i}}^{3} \frac{z - z_l}{z_i - z_l}, \quad i = 0, \dots, 3$$

$$\Rightarrow y(x_{k+4}) = y(x_{k+3}) + \sum_{i=0}^{m=3} b_i f_{k+i}$$

$$b_i = \int_{x_{k+3}}^{x_{k+4}} L_{k+i}(x)\, dx$$



$$\Rightarrow b_i = \int_{3h}^{4h} L_i(x)\, dx$$

$$\Rightarrow b_0 = \int_{3h}^{4h} \prod_{\substack{l=0 \\ l \neq 0}}^{3} \frac{x - x_{l+k}}{x_{0+k} - x_{l+k}}\, dx = \int_{3h}^{4h} \frac{x-h}{0-h} \frac{x-2h}{0-2h} \frac{x-3h}{0-3h}\, dx$$

$$= \int_{3h}^{4h} 1 - \frac{11}{6h} x + \frac{x^2}{h^2} - \frac{x^3}{6h^3}\, dx = h - \frac{11h^2}{12h}(16-9) + \frac{h^3}{3h^2}(4^3 - 3^3) - \frac{h^4}{24h^3}(4^4 - 3^4)$$

$$= -\frac{3}{8} h$$

$$b_1 = \int_{3h}^{4h} \prod_{\substack{l=0 \\ l \neq 1}}^{3} \frac{x - x_{l+k}}{x_{1+k} - x_{l+k}} = \int_{3h}^{4h} \frac{x-0}{h-0} \frac{x-2h}{h-2h} \frac{x-3h}{h-3h}\, dx = \frac{37h}{24}$$

$$b_2 = \int_{3h}^{4h} \prod_{\substack{l=0 \\ l \neq 2}}^{3} \frac{x - x_{l+k}}{x_{2+k} - x_{l+k}} = \int_{3h}^{4h} \frac{x-0}{2h-0} \frac{x-h}{2h-h} \frac{x-3h}{2h-3h}\, dx = -\frac{59}{24} h$$

$$b_3 = \int_{3h}^{4h} \prod_{\substack{l=0 \\ l \neq 3}}^{3} \frac{x - x_{l+k}}{x_{3+k} - x_{l+k}} = \int_{3h}^{4h} \frac{x-0}{3h-0} \frac{x-h}{3h-h} \frac{x-2h}{3h-2h}\, dx = \frac{55}{24} h$$

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Aufgabe 3

Kollokationsverfahren:

Sind $c_1, \dots c_s$ NS von Legendre-Polynom Grad s, dann hat RK-Verfahren Ordnung 2s

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$\int_0^1 P_2(x)\,dx = 0$

$P_2(x) \overset{!}{=} 0 \iff x_1 = -\frac{1}{\sqrt{3}},\ x_2 = \frac{1}{\sqrt{3}} \Rightarrow c_1 = x_1,\ c_2 = x_2$

$l_1(x) = \prod_{\substack{j=1 \\ j \neq 1}}^{2} \frac{x - c_j}{c_i - c_j} = \frac{1}{2}(1 - \sqrt{3}x)$

$l_2(x) = \frac{x - c_1}{c_2 - c_1} = \frac{1}{2}(1 + \sqrt{3}x)$

$b_1 = \int_0^1 l_1(x)\,dx = \frac{1}{2} - \frac{\sqrt{3}}{4}$

$b_2 = \int_0^1 l_2(x)\,dx = \frac{1}{2} + \frac{\sqrt{3}}{4}$

$a_{11} = -\frac{1}{4\sqrt{3}}$, $a_{ij} = \int_0^{c_i} l_j(t)\,dt$

$a_{21} = -\frac{\sqrt{3}}{4}$

$a_{12} = \frac{\sqrt{3}}{4}$

$a_{22} = \frac{1}{4\sqrt{3}}$

$$\Rightarrow \begin{array}{c|cc} -\frac{1}{\sqrt{3}} & -\frac{\sqrt{3}}{4} & -\frac{1}{4\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{4\sqrt{3}} & \frac{\sqrt{3}}{4} \\ \hline & \frac{1}{2} - \frac{\sqrt{3}}{4} & \frac{1}{2} + \frac{\sqrt{3}}{4} \end{array}$$

$\Rightarrow\ k_1 = f\left(x_k - \frac{1}{\sqrt{3}}h,\ y_k - \frac{1}{4}\left(\frac{k_1}{\sqrt{3}} + k_2\sqrt{3}\right)h\right)$

$k_2 = f\left(x_k + \frac{1}{\sqrt{3}}h,\ y_k + \frac{1}{4}\left(\sqrt{3}k_1 + k_2\frac{1}{\sqrt{3}}\right)h\right)$

$y_{k+1} = y_k + h\left(\left(\frac{1}{2} + \frac{\sqrt{3}}{4}\right)k_1 + \left(\frac{1}{2} - \frac{\sqrt{3}}{4}\right)k_2\right)$