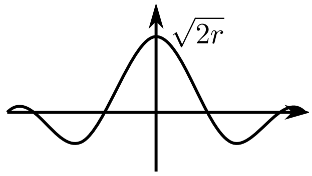
9 Übungsblatt von Analysis 4 zum Mittwoch, den 8.6.2011

Aufabe 1

a) $\psi(x) = \int\limits_{\mathbb{R}} \frac{1}{\sqrt{2r}} \mathbb{1}_{[-r,r]}(k) e^{ikx} dk = \begin{cases} \sqrt{2r}\frac{\sin(rx)}{rx} & , x \neq 0 \\ \sqrt{2r} & , x = 0 \end{cases}$



b) Zu prüfen: $\int\limits_{\mathbb{R}} |\psi(x)|^2 dx ex.$

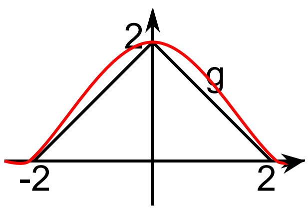
d.h. $\lim\limits_{R\to\infty} \int\limits_{-R}^{R} |\psi(x)|^2 dx < \infty$

$(\frac{\sin(rx)}{rx})^2 \leq \underbrace{\mathbb{1}_{[-R,R]}(x)\sqrt{2r}}_{(x\mapsto[-R,R])\in\mathbb{L}^1} + \underbrace{\mathbb{1}_{\mathbb{R}\setminus[-R,R]}\frac{1}{r^2x^2}}_{(x\mapsto\mathbb{R}\setminus[-R,R])\in\mathbb{L}^1}$

$\Rightarrow (x \mapsto (\frac{\sin(rx)}{rx})^2) \in \mathbb{L}^1$

c) $\Delta k = r \quad \Rightarrow \Delta k \Delta x = r\frac{\pi}{r} = \pi$

Aufabe 2



$f = \mathbb{1}_{[-1,1]}, \quad \hat{f}(x) = \sqrt{\frac{2}{\pi}}\frac{\sin(x)}{x}$

$\int\limits_{\mathbb{R}} \frac{\sin^2(x)}{x^2} dx = \frac{\pi}{2} \int\limits_{\mathbb{R}} (\hat{f}(x))^2 dx \overset{Satz\ 8.3b)}{=} \frac{1}{\sqrt{2\pi}}\frac{\pi}{2} \int\limits_{\mathbb{R}} \widehat{(f*f)}(x) dx$

$(f*f)(x) := \begin{cases} 2-x & , x \in [0,2] \\ 2+x & , x \in [-2,0] \\ 0 & , sonst \end{cases}$

$\int\limits_{\mathbb{R}} \frac{\sin^2(x)}{x^2} dx = \frac{\pi}{2}\frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} \hat{g}(x) dx = \frac{\pi}{2}(\hat{g})^{\vee}(0)$

$= \frac{\pi}{2} g(0) = 2\frac{\pi}{2} = \pi$

Aufabe 3

Beh.: $\hat{\mathbb{F}}\psi = \mathbb{F}\hat{H}\psi \forall \psi \in S$

$\hat{\mathbb{F}}\psi = -\frac{1}{2}\hat{\psi}''(x) + \frac{1}{2}x^2\hat{\psi}(x)$

$\hat{\psi}(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} \psi(k) e^{-ikx} dk \overset{\psi(R) \underset{|R|\to\infty}{\longrightarrow} 0}{=} -\frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} \psi'(x)(-\frac{1}{ix}) e^{-ikx} dk$

$\overset{\psi'(x) \underset{|x|\to\infty}{\longrightarrow} 0}{=} \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} psi''(k)(-\frac{1}{ix})^2 e^{-ikx} dk$

$\Rightarrow -x^2\hat{\psi}(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} \psi''(k) e^{-ikx} dk = (\widehat{\psi''(x)})(x)$

$\hat{H}\mathbb{F}\psi(x) = -\frac{1}{2}\hat{\psi}''(x) + \underbrace{\frac{1}{2}x^2\hat{\psi}(x)}_{-(\widehat{\psi''})(x)} = -\frac{1}{2}(\hat{\psi}''(x) - (\widehat{\psi''})(x))$

$\mathbb{F}\hat{H}\psi(x) = \mathbb{F}(-\frac{1}{2}\psi''(x) + \frac{1}{2}x^2\psi(x)) = -\frac{1}{2}(\widehat{\psi''})(x) + \frac{1}{2}(\widehat{x^2\psi(x)})$

$(\widehat{x^2\psi(x)}) = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} k^2\psi(k) e^{-ikx} dx$

$= -\frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} (2k\psi(x) + k^2\psi'(k))\frac{e^{-ikx}}{(-ix)} dx$

$\hbar\hat{\psi}'(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} \psi(k)(-ix)e^{-ikx}dk$
$\hat{\psi}''(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} \psi(k)(-ix)^2 e^{-ikx}dk$
$= -x^2 \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} \psi(k) e^{-ikx}dx = -x^2\hat{\psi}(x)$
Rest nachgeliefert!

Aufabe 4

c) $\forall n \in \mathbb{N}: \; H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x) \quad \Rightarrow (a),(b)$
$\Rightarrow H_n''(x) = 2H_n(x) + 2xH_n'(x) - H_{n+1}'(x)$
$= 2H_n(x) + 2x(2xH_n(x) - H_{n+1}(x)) - 2xH_{n+1}(x) + H_{n+2}(x)$
$= (2+4x^2)H_n(x) - 4xH_{n+1}(x) + H_{n+2}(x)$
$h_n'(x) = (-x)e^{-\frac{x^2}{2}}H_n(x) + e^{-\frac{x^2}{2}} = (-x)h_n(x) + e^{-\frac{x^2}{2}}H_n'(x)$
$h_n''(x) = -h_n(x) + (-x)h_n'(x) + (-x)e^{-\frac{x^2}{2}}H_n'(x) + e^{-\frac{x^2}{2}}H_n''(x)$
$= -h_n(x) + (-x)^2h_n(x) + 2(-x)e^{-\frac{x^2}{2}}H_n'(x) + e^{-\frac{x^2}{2}}H_n''(x)$
$= -h_n(x) + x^2h_n(x) - 2xe^{-\frac{x^2}{2}}(2xH_n(x) - H_{n+1}(x)) + e^{-\frac{x^2}{2}}((2+4x^2)H_n(x) - 4xH_{n+1}(x) + H_{n+2}(x))$
$= -h_n(x) + x^2h_n(x) - 4x^2h_n(x) + 2xh_{n+1}(x) + 2h_n(x) + 4x^2h_n(x) - 2xh_{n+1}(x) + h_{n+2}(x)$
$= (1+x^2)h_n(x) - 2xh_{n+1}(x) + h_{n+2}(x)$

d) $\hat{H}h_n(x) = -\frac{1}{2}h_n''(x) + \frac{1}{2}x^2h_n(x) = -\frac{1}{2}((1+x^2)h_n(x)9 - 2xh_{n+1}(x) + h_{n+2}(x)) + \frac{1}{2}x^2h_n(x)$
$-\frac{1}{2}(h_n(x) - 2xh_{n+1}(x) + \underbrace{h_{n+2}(x)}_{2xh_{n+1}(x)-2(n+1)h_n(x)})$
$= -\frac{1}{2}(h_n(x) - 2(n+1)h_n(x)) = \frac{2n+1}{2}h_n(x)$

e) $< \hat{H}h_n, h_m >_{L^2} = < h_n, \hat{H}h_m >_{L^2}$
$< \hat{H}h_n, h_m >_{L^2} = \int\limits_{\mathbb{R}} (\hat{H}h_n)h_m dx = \int\limits_{\mathbb{R}} (-\frac{1}{2}h_n''(x) + \frac{1}{2}h_n(x))h_m dx$
$(|x| \to \infty, \; h_m \to 0, \; h_n' \to 0)$
$= \int\limits_{\mathbb{R}} (-\frac{1}{2}h_n(x)h_m''(x) + \frac{1}{2}h_n(x)h_m(x))dx$
$= \int\limits_{\mathbb{R}} h_n(x)(\hat{H}h_m(x))dx = < h_n, \hat{H}h_m >_{L^2}$
$\frac{2n+1}{2} < h_n, h_m >_{L^2} = < \hat{H}h_n, h_m >_{L^2} = < h_n, \hat{H}h_m >_{L^2}) \frac{2m+1}{2} < h_n, h_m >_{2L}$

$m \neq m \quad \Rightarrow < h_n, h_m >_{L^2} = 0$

Aufgabe 5

a) $k_\varepsilon(\omega) = \frac{1}{\pi}\frac{\varepsilon}{\varepsilon^2 + (\omega - \omega_0)^2}$
b) $f \in C_c^\infty(\mathbb{R}), \quad |k_\varepsilon(\omega)| \leq \frac{\varepsilon}{\varepsilon^2} = \frac{1}{\varepsilon}$
$|fk_\varepsilon| \leq \frac{1}{\varepsilon}f \in \mathbb{L}^1$
$f \cdot k_\varepsilon mbar \quad \Rightarrow fk_\varepsilon \in \mathbb{L}^1$

$\Rightarrow I_\varepsilon = \int\limits_{\mathbb{R}} f k_\varepsilon d\omega = \int\limits_{\mathbb{R}} f d\alpha_\varepsilon(\omega)$

$$\alpha_\varepsilon(x) = \frac{1}{\pi}\arctan(\frac{\omega-\omega_2}{\varepsilon}) \underset{\varepsilon \to 0}{\longrightarrow} \begin{cases} 0 & ,\omega = \omega_0 \\ \frac{1}{2} & ,\omega > \omega_0 \\ -\frac{1}{2} & ,\omega < \omega_0 \end{cases}$$

c) Sei $\rho > 0$. $\exists \delta_1 > 0$ mit $|f(\omega) - f(\omega_0)| < \frac{\rho}{2} \;\; \forall \omega \in [\omega_0 - \delta_1, \omega_0 + \delta_1]$
$\exists \varepsilon_0 > 0$ mit $\forall \varepsilon < \varepsilon_0 : \;\; |\frac{1}{\pi}\arctan(\frac{\omega}{\varepsilon}) - \frac{1}{2}| < \frac{\varepsilon}{2} \;\; \forall \omega \in \mathbb{R},\; |\omega| \geq \delta_1$
Dann ist $(t_i)_I$ eine Zerlegung von $\mathbb{R}$. o.E. $t_i \neq \omega_0 \forall i \in I$ mit Feinheit kleiner als $\frac{\delta_1}{2}$
Sei $j \in I$ mit $t_{j-1} < \omega_0 < t_j$

$$|\sum_{i\in I} f(\xi)(\alpha(t_i)\alpha(t_j)) - f(\omega_0)| \underset{Feinheit \to 0}{\longrightarrow} I_\varepsilon \begin{cases} \underset{\varepsilon \to 0}{\longrightarrow} f(\omega_0) \\ \longrightarrow I = f(\omega_0) \end{cases}$$

$\leq \sum\limits_{i\in I} |f(\xi_i)(\alpha(t_i) - \alpha(t_j)) - f(\omega_0)| + 2\frac{\rho}{2}$
$(|\frac{t_i - \omega_0}{\varepsilon}| \leq \delta_1)$
$\leq \sum |f(\xi_i) - f(\omega_0)|(\alpha(t_i) - \alpha(t_{i-1}) \leq \sup\limits_{i\in I} |f(\xi_i) - f(\omega_0)| \sum(...) + \rho$
$\leq \frac{\rho}{2} 2\delta_1 + \rho = (\delta_1 + 1)\rho \underset{\rho \to 0}{\longrightarrow} 0$ (unabgh. Zerlegung)
$\Rightarrow |I_\varepsilon - f(\omega_0)| \leq (\delta_1 + 1)\rho \;\; \forall \varepsilon < \varepsilon_0$