Klausur Theo 4 SS10 Lenske 2

K2

a) $f_1 = \frac{1}{2i\pi}\frac{e^{ixz}}{z-i\varepsilon} \Rightarrow$ Polstelle $z_1 = i\varepsilon$ $\quad f_2 = \frac{1}{2i\pi}\frac{e^{-ixz}}{z-i\varepsilon} \Rightarrow$ Polstelle $z_2 = i\varepsilon$

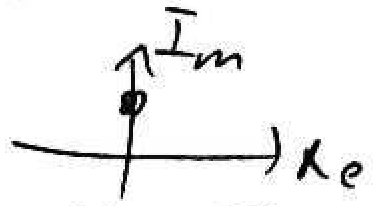
$\mathrm{Res}_{f_1} = \frac{1}{2\pi i}\frac{e^{ixz}}{1}\Big|_{z=i\varepsilon} = \frac{1}{2\pi i}e^{-\varepsilon x} \xrightarrow[\varepsilon\to 0^+]{} 1$

$\mathrm{Res}_{f_2} = \frac{1}{2\pi i}\frac{e^{-ixz}}{1}\Big|_{z=i\varepsilon} = \frac{1}{2\pi i}e^{\varepsilon x} \xrightarrow[\varepsilon\to 0^+]{} 1$

~~e-fu: $\frac{1}{\alpha z}$ $\leftarrow$ Grad 0, $\leftarrow$ Grad 1 $\Rightarrow \int_{-\infty}^{\infty} \ldots = 2\pi i \sum_i \mathrm{Res}_f(1)$~~

$= s(x) = e^{-\varepsilon x} - e^{\varepsilon x} \xrightarrow[\varepsilon\to 0^+]{} 1 - 1 = 0$

b) Polstelle nur im + Im $\Rightarrow$



$\Rightarrow$ Weg über Pos. Im-Ebene mit $\curvearrowleft_R$, $R\to\infty$

$x \geq 0 \Rightarrow$ pos Halbebene, $x < 0 \Rightarrow$ neg. H-Ebene

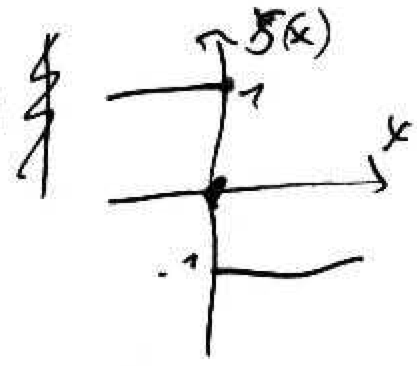
Mathe: efu: grad oben $\geq$ grad unten + 1

$\Rightarrow \int_{-\infty}^{\infty} = \sum 2\pi i \,\mathrm{Res}_f(1)$

$s(x) = \frac{1}{2i\pi}\begin{cases}\int_{-\infty}^{\infty} \frac{e^{ixz}}{z-i\varepsilon}dz & x \geq 0\\ \int_{\infty}^{-\infty} \frac{e^{ixz}}{z-i\varepsilon}dz & x < 0\end{cases} - \frac{1}{2i\pi}\begin{cases}\int_{\infty}^{-\infty} \frac{e^{-ixz}}{z-i\varepsilon}dz & x \geq 0\\ \int_{-\infty}^{\infty} \frac{e^{ixz}}{z-i\varepsilon}dz & x < 0\end{cases}$

$\Rightarrow$ $x > 0$: 2. Integral untere ImRe-Ebene $\Rightarrow = 0 + 1$
$x < 0$: 1. Integral " $\Rightarrow = 0 + 1$
$x = 0$: beide Integrale umschließen Res nicht $\Rightarrow 0$

c) $\Rightarrow$ $s(x) = \begin{cases} e^{-\varepsilon x} - 0 & x > 0\\ 0 & x = 0 \\ 0 - e^{\varepsilon x} & x < 0\end{cases} \longrightarrow \begin{matrix}1\\0\\-1\end{matrix} \Rightarrow$



K3

a) $A = \begin{pmatrix}\cos(\alpha) & b\,e^{i\gamma}\sin(\alpha)\\ a\,e^{-i\gamma}\sin(\alpha) & \cos(\alpha)\end{pmatrix} \Rightarrow A^+ = \begin{pmatrix}\cos(\alpha) & a^*e^{i\gamma}\sin(\alpha)\\ b^*e^{-i\gamma}\sin(\alpha) & \cos(\alpha)\end{pmatrix}$

hermitisch: $A = A^+ \Rightarrow a^* = b,\ b^* = a \Rightarrow \mathrm{Re}(a) = \mathrm{Re}(b)$, $\mathrm{Im}(a) = -\mathrm{Im}(b)$

b) unitär: $AA^+ = \mathbb{1} \Rightarrow \begin{pmatrix}\cos^2(\alpha) + |b|\sin^2(\alpha) & \sin(\alpha)\cos(\alpha)e^{i\gamma}(a^*+b)\\ \sin(\alpha)\cos(\alpha)e^{-i\gamma}(a+b^*) & \cos^2 + |a|\sin^2(\alpha)\end{pmatrix}$

$\Rightarrow |a| = |b| = 1,\ a + b^* = a^* + b = 0 \Rightarrow a = \frac{-1}{b}$

c) orthogonal: $A^T = A^{-1} \Rightarrow A^TA = \mathbb{1} \Rightarrow A^T \stackrel{!}{=} A^+$

$\Rightarrow a^*e^{i\gamma} = a\,e^{-i\gamma} \Leftrightarrow a = -b\,e^{2i\gamma} \Rightarrow a_1 = -\frac{1}{-a}e^{2i\gamma} = e^{i\gamma} \Rightarrow b_1 = -e^{-i\gamma}$

$b^*e^{-i\gamma} = b\,e^{i\gamma} \Leftrightarrow b = -a\,e^{2i\gamma}$ $\quad$ ~~$b = -\frac{1}{-b}e^{2i\gamma} = e^{i\gamma}$~~

$-1 = -b^2e^{2i\gamma} \Leftrightarrow b_2 = e^{-i\gamma} \Rightarrow a_2 = -e^{i\gamma}$

$\Rightarrow a_n = \pm e^{i\gamma},\ b = \mp e^{-i\gamma}$

d) EW: $\det(A - \mathbb{1}\lambda) = (\cos(\alpha) - \lambda)^2 - (ab\, \sin^2(\alpha))$

$= \cos^2(\alpha) - 2\cos(\alpha)\lambda + \lambda^2 - ab\sin^2(\alpha)$

$= \cos(\alpha) \pm \sqrt{\cos^2(\alpha) - \cos^2(\alpha) + ab\sin^2(\alpha)} = \cos(\alpha) \pm \sin(\alpha)\sqrt{ab}$

unitär: $\sqrt{ab} = \sqrt{-1} = i \Rightarrow \lambda_{1,2} = e^{\mp i\alpha}$

e) ~~$\lambda_1 = e^{i\alpha}$~~ : ~~$\begin{pmatrix} \cos\alpha - e^{i\alpha} & b e^{i\gamma}\sin(\alpha) \\ a e^{-i\gamma}\sin(\alpha) & \end{pmatrix}$~~

$\lambda_1 = \cos(\alpha) + \sin(\alpha)\sqrt{ba}$: $\begin{pmatrix} \cos(\alpha) - \cos(\alpha) - \sin(\alpha)\sqrt{ba} & b e^{i\gamma}\sin\alpha \\ a e^{-i\gamma}\sin(\alpha) & \cos\alpha - \cos\alpha - \sin(\alpha)\sqrt{ba} \end{pmatrix}$

$\sim \sin(\alpha)\begin{pmatrix} -\sqrt{ba} & b e^{i\gamma} \\ a e^{-i\gamma} & -\sqrt{ba} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} -\sqrt{ba}\,x + b e^{i\gamma} y = 0 \\ a e^{-i\gamma} x - \sqrt{ba}\, y = 0 \end{matrix}$

$\Rightarrow x = \frac{b}{\sqrt{ba}} e^{i\gamma} y$ ~~$\Rightarrow y = \frac{a}{\sqrt{ba}} e^{-i\gamma} \cdot \frac{b}{\sqrt{ba}} e^{i\gamma} y$~~ $\Rightarrow \begin{pmatrix} \frac{b}{\sqrt{ba}} e^{i\gamma} \\ 1 \end{pmatrix} \cdot n = v_1$

$\lambda_2 = \cos(\alpha) - \sin(\alpha)\sqrt{ba}$: $\sin(\alpha)\begin{pmatrix} \sqrt{ba} & b e^{i\gamma} \\ a e^{-i\gamma} & \sqrt{ba} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow x = -\frac{b}{\sqrt{ba}} e^{i\gamma} \Rightarrow v_2 = \begin{pmatrix} -\frac{b}{\sqrt{ba}} e^{i\gamma} \\ 1 \end{pmatrix} \cdot n$

norm $n : \frac{1}{|(\dots)|} = \sqrt{1 + \left|\frac{b}{\sqrt{ba}}\right|^2 \cdot |e^{i\gamma}|^2}^{-1} = \sqrt{1 + \frac{b^2}{ba}}^{-1} = \sqrt{1 + \frac{b}{a}}^{-1}$