$\sin^4(x)$ Fourierreihe?

$$\sum_{k\in\mathbb{Z}} \langle e^{ikx} | f\rangle e^{ikx} \cdot \frac{1}{2\pi}$$

$$\sum \frac{1}{\pi}\left(\langle \sin(kx)|f\rangle \sin(kx) + \langle\cos(kx)|f\rangle \cos(kx)\right) + \langle \tfrac{1}{\sqrt{2}} | f\rangle \frac{1}{\sqrt{2}\pi}$$

$$\sum a_n z^n \qquad a_n = \frac{1}{2\pi i}\oint \frac{f(z)}{z^{n+1}}\,dz$$

$$\sin^4(x) = \left(\frac{e^{ix}-e^{-ix}}{2i}\right)^4 = \frac{1}{(2i)^4}\left(e^{4ix} - 4e^{3ix}e^{-ix} + 6e^{2ix}e^{-2ix} - 4e^{ix}e^{-3ix} + e^{-4ix}\right)$$

$$\sin^4(x) = \frac{1}{16}\left(e^{4ix} - 4e^{2ix} + 6 - 4e^{-2ix} + e^{-4ix}\right)$$

$$= \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

$$= \frac{1}{16}\left(2\cos(4x) - 4\cdot 2\cos(2x) + 6\right)$$

$$= \frac{6}{16}\cdot 1 - \frac{4\cdot 2}{16}\cos(2x) + \frac{2}{16}\cos(4x)$$

$\frac{1}{\pi}\langle\cos(4x)|\sin^4(x)\rangle$

$$\langle \sin^4(x) | \sin(kx)\rangle$$

$$\sum \underbrace{a_k}_{\langle\varphi_k|f\rangle} \underbrace{\varphi_k}_{\text{orthogonal!}}$$

$$[0,4\pi] \quad e^{i\frac{k}{2}x}$$

$$\frac{1}{x^2\sin(x)+e^x}\mathbb{1}_{[0,\infty)} \quad \text{messbar}$$

f messbar $\Leftarrow$ f stetig, Ableitung von differenzierbarer Funktion, f monoton, Indikator von messbarer Menge

$$\int \underbrace{\underbrace{\left|\frac{1}{x^2\sin^2 x+e^x}\right|}_{\geq 0}\mathbb{1}_{[0,\infty)}}_{\leq \frac{1}{e^x}\mathbb{1}_{[0,\infty)}}\,d\lambda \leq \int_0^\infty e^{-x}\,dx = 1 < \infty$$

$$\left\|\frac{1}{x^2\sin^2 x+e^x}\right\|_\infty \leq \|e^{-x}\|_\infty = 1 < \infty$$

$\text{ess-sup } \mathbb{1}_{\mathbb{Q}} = 0 \qquad \mathbb{1}_{\mathbb{Q}} = 0$ f.ü.

$\text{ess-sup } f = \inf_{\lambda(N^c)=0} \sup_{x \in M} f(x)$

$\lambda(\underbrace{\{x \in \mathbb{R} \mid \mathbb{1}_{\mathbb{Q}}(x) = 1\}}_{\mathbb{1}_{\mathbb{Q}}^{-1}(1)}) = \lambda(\mathbb{Q}) = 0$

$\lambda(\mathbb{1}_{\mathbb{Q}}^{-1}\{0\}) = \lambda(\mathbb{R} \setminus \mathbb{Q}) = \lambda(\mathbb{R}) - \lambda(\mathbb{Q}) = \infty - 0 = \infty$

$\sup(\mathbb{1}_{\mathbb{Q}}) = 1 \qquad \|\mathbb{1}_{\mathbb{Q}}\|_{\infty}^{\text{esssup}} = 0$

$f(x) = \left\{\begin{matrix} 0 & \forall x \in \mathbb{R} \setminus \{0\} \\ \infty & x = 0 \end{matrix}\right\} = 0$ f.ü.

$\sin x \in \mathcal{L}_1(\mathbb{R})$? Nein, denn angenommen $\sin(x) \in \mathcal{L}^1$

$\Rightarrow \cos(x) = \sin(x - \frac{\pi}{2}) \in \mathcal{L}^1$

$\Rightarrow e^{ix} = \cos(x) + i \sin(x) \in \mathcal{L}^1$

$\int |e^{ix}| \, dx = \infty$ ↯

$\sin x \geq \frac{1}{2} \quad \forall x \in [\frac{\pi}{4}, \frac{3}{4}\pi]$

$\int |\sin x| \, dx \geq \sum_{k \in \mathbb{Z}} \int_{[\frac{1}{4}\pi, \frac{3}{4}\pi] + k\pi} \underbrace{|\sin x|}_{\geq \frac{1}{2}} \, dx = \infty$

$\log|x| \, e^{-|x|} \qquad x = 0$ Problem? Nein, denn:

$\int_1^{\infty} \frac{1}{x} \, dx$ Pfui!

Positiv: okay! $\int |\log|x| \, e^{-|x|}| \, dx = \underbrace{2 \int_0^1 |\log|x|| \underbrace{e^{-|x|}}_{\leq 1}}_{\leq \int_0^1 |\log|x|| \, dx} + \underbrace{2 \int_1^{\infty} \underbrace{|\log|x||}_{\leq x} e^{-|x|} \, dx}_{< \infty}$

$= -(\underbrace{x \log x}_{\xrightarrow{x \to 0} 0} - x)\Big|_0^1 = 1 < \infty$

$\mathcal{L}^1, \mathcal{L}^2, \mathcal{L}^\infty$

$f \in \mathcal{L}^\infty : \exists c > 0 \mid |f| \le c$ f.ü.

$\frac{1}{x} \notin \mathcal{L}^\infty(\mathbb{R})$, denn $\forall c > 0 : |\frac{1}{x}| \ge c \quad \forall x \in [-\frac{1}{c}, \frac{1}{c}]$

$\lambda(\dots) = \frac{2}{c} > 0$

$\underbrace{\frac{1}{x} \mathbb{1}_{[-\varepsilon,\varepsilon]^c}} \in \mathcal{L}^\infty$

$\|f\|_1 = \underbrace{\int_{-\infty}^{-\varepsilon} |\frac{1}{x}| dx}_{=\infty} + \underbrace{\int_{\varepsilon}^{\infty} |\frac{1}{x}| dx}_{=\infty} = \infty \Rightarrow f \notin \mathcal{L}^1$

$\frac{1}{x+x^3} \notin \mathcal{L}^1 \qquad \frac{1}{x+x^3} \mathbb{1}_{(1,\infty)} \in \mathcal{L}^1$

$\int_0^1 \frac{1}{\underbrace{x+x^3}_{\le 2x}} \ge \frac{1}{2} \int_0^1 \frac{1}{x} dx = \infty$

$\frac{1}{\sqrt{x}} \mathbb{1}_{[0,1]} \begin{array}{l} \in \mathcal{L}^1 \\ \notin \mathcal{L}^2 \end{array} \Rightarrow$ Singularitäten um Null!!!

$f_n \xrightarrow{\text{f.ü.}} f \overset{?}{\Rightarrow} \int f_n \to \int f$

falls $|f_n| \le g \in \mathcal{L}^1$

$\underbrace{e^{-\frac{|x|}{n}}}_{\xrightarrow[n\to\infty]{} 1} \mathbb{1}_{[0,1]} \qquad |e^{-\frac{|x|}{n}}| \le 1 \in \mathcal{L}^1[0,1]$

$\Rightarrow \int_0^1 e^{-\frac{|x|}{n}} \to \int_0^1 1 = 1$

$\underbrace{\int_0^\infty e^{-x/n} dx}_{n \int_0^\infty e^{-x} dx = n} \xrightarrow[n\to\infty]{\text{BEPPO LEVI}} \int_0^\infty 1 \, dx = \infty$

$$\int_{\mathbb{R}} \underbrace{\cos\left(\frac{x}{n}\right)}_{\xrightarrow{n\to\infty} 1} \cdot \frac{1}{x^2+1}\,dx \xrightarrow{\text{Lebesgue}} \int_{\mathbb{R}} 1 \cdot \frac{1}{x^2+1}\,dx = \pi$$

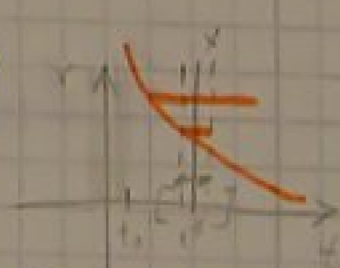
$$\Big|\underbrace{\cos\left(\frac{x}{n}\right)}_{|\,|\le 1} \cdot \frac{1}{x^2+1}\Big| \le \frac{1}{x^2+1} \in \mathcal{L}^1 \quad \Big(\int_{\mathbb{R}} \Big|\frac{1}{x^2+1}\Big| \le \int_{[-1,1]} 1 + \int_1^\infty \frac{1}{x^2}\,dx + \int_{-\infty}^{-1} \frac{1}{x^2}\,dx < \infty\Big)$$

$$\int_{\mathbb{R}} \frac{dx}{x^2} \ge \int_0^1 \frac{dx}{x^2} = -\frac{1}{x}\Big|_0^1 \quad ↯ \quad \frac{1}{x} \notin \mathcal{L}^2(\mathbb{R})$$

$$\neg\exists \int_{\mathbb{R}} \frac{1}{x^2}\,dx \ !$$

$$\partial_x \int \underbrace{\cos(xy)}_{\in C^1 \,|\, \partial_x \cos(xy) = -y\sin(xy)} \cdot \frac{1}{y^4+1}\,dy = \int \partial_x \cos(xy) \frac{1}{y^4+1}\,dy = -\int \sin(xy) \frac{y}{y^4+1}\,dy$$

$$\Big|\frac{y\sin(xy)}{y^4+1}\Big| \le \frac{y}{y^4+1} \in \mathcal{L}^1$$ unabhängig von x!



$\forall t \ge t_0 > 0$: $$\partial_t \int_1^\infty \frac{e^{-xt}}{x}\,dx \qquad \Big|\partial_t \frac{e^{-xt}}{x}\Big| = |-e^{-xt}| \le e^{-t_0 x} \in \mathcal{L}^1 \quad \forall t \ge t_0$$

$$= \int_1^\infty \partial_t \frac{e^{-xt}}{x}\,dx = \int_1^\infty -e^{-xt}\,dx = \frac{1}{t} e^{-xt}\Big|_1^\infty = -\frac{1}{t}e^{-t}$$

Plancherel: $\|\hat f\|_2 = \|f\|_2$

$\hat f(x) = \frac{1}{\sqrt{2\pi}} \int f(t) e^{-ixt}\,dt \quad \forall f \in \mathcal{L}^1$

$\forall f \in \mathcal{L}^2$: $f \underbrace{\mathbb{1}_{[-n,n]}}_{\in \mathcal{L}^2} \in \mathcal{L}^1$ $\qquad \|fg\|_1 \le \|f\|_2 \|g\|_2$!

$\Rightarrow \underbrace{f}_{\in\mathcal{L}^2} \underbrace{\mathbb{1}_{[-n,n]}}_{\in\mathcal{L}^\infty} \Rightarrow \in \mathcal{L}^2$ hat Fouriertransformierte $\widehat{f\mathbb{1}_{[-n,n]}}$

$f\mathbb{1}_{[-n,n]} \xrightarrow{\text{f.ü.}} f$

$$|f^2 \mathbb{1}^2_{[-n,n]}| \leq |f^2| \in \mathcal{L}^1 \qquad (f \in \mathcal{L}^2)$$

$$\Rightarrow \underbrace{f \mathbb{1}_{[-n,n]}}_{f_n} \xrightarrow{\mathcal{L}^2} f$$

$$\Rightarrow \|\hat{f}_n - \hat{f}\|_2 = \|\widehat{f_n - f}\|_2 = \|f_n - f\|_2 \longrightarrow 0$$

Konvergenz in $\mathcal{L}^2$ durch Satz von Plancherel

$$\frac{1}{\sqrt{2\pi}} \int_{-n}^{n} f(t) e^{-ixt}\, dt \xrightarrow[n\to\infty]{\mathcal{L}^2} \hat{f}$$