Ana 4 Klausur Lani 2006

Nr 1

a) $(u_j) \in \mathcal{L}^1$, $u_j \to u$ f.ü.,
$\exists f \in \mathcal{L}^1 : \forall n \in \mathbb{N} : |u_j| \leq f \Rightarrow u \in \mathcal{L}^1$ und $\int u_j \to \int u$

b) $f_n(x) = n^2 \mathbb{1}_{(0,\frac{1}{n})}$
$\int f_n(x)\,dx = n^2 \cdot \frac{1}{n} = n \xrightarrow[n\to\infty]{} \infty$
$f_n(x) = \begin{cases} 0 & , x \notin (0,\frac{1}{n}) \\ n^2 & , x \in (0,\frac{1}{n}) \end{cases} \Rightarrow f_n(x) = 0 \quad \forall x \in (-\infty,0) \cup (1,\infty)$
$\forall x \in [0,1] \; \exists n_0 \in \mathbb{N} : n_0 > \frac{1}{x} \Rightarrow x \notin (0,\frac{1}{n_0})$
$\Rightarrow f_n(x)\big|_{x\in[0,1]} \xrightarrow[n\to\infty]{} 0$

c) $\cancel{\frac{1}{1+x^2} \leq 1} \Rightarrow \cancel{f_n(x) = (n^2+1)\mathbb{1}_{(0,1)}}$
$\cancel{f_n(x) = -\frac{n^2}{x^2}\mathbb{1}_{(0,\frac{1}{n})}} \Rightarrow \cancel{\int f_n(x)\,dx = -n^2\left[-\frac{1}{x^2}\right]_0^{\frac{1}{n}}} =$
$f_n(x) = \frac{n^2}{1+x}\mathbb{1}_{(0,\frac{1}{n})} \Rightarrow \int f_n(x)\,dx = n^2\left[\ln(x+1)\right]_0^{\frac{1}{n}}$
$= n^2 \ln(\frac{1}{n}) \xrightarrow[n\to\infty]{} \infty$
$\cancel{f_n(x)}$
$f_n(x) \xrightarrow[n\to\infty]{} 0$ (siehe b))

Nr 2

a) $(u_j) \in \mathcal{L}^1$ monoton wachsend/fallend, $\int(u_j)$ beschr.
$\Rightarrow \exists u \in \mathcal{L}^1 : u_j \to u$ f.ü. und $\int u_j \to \int u$

b) $\frac{1}{2\sqrt{x}+x^2} \leq \frac{1}{2\sqrt{x}} \leq \frac{1}{\sqrt{x}} \quad \forall x \in (0,\infty)$
$\int \frac{1}{\sqrt{x}} = \cancel{\ } - \frac{1}{\sqrt{x^3}} \Rightarrow$ Integrable Majorante von $f = \frac{1}{\sqrt{x}} \Rightarrow f \in \mathcal{L}^1$

Nr 3

a) $\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{-iwx}x\,dx = \frac{1}{\sqrt{2\pi}}\left(\left[-\frac{x}{iw}e^{-iwx}\right]_{-1}^{1} + \frac{1}{iw}\int_{-1}^{1} e^{-iwx}\,dx\right)$
$= \frac{1}{\sqrt{2\pi}}\left(2i\frac{1}{w}\cos(w) + \left[\frac{1}{-iw^2}e^{-iwx}\right]_{-1}^{1}\right)$
$= \frac{1}{\sqrt{2\pi}}\left(2i\cdot\frac{1}{w}\cos(w) + \frac{1}{w^2}\cdot 2i\sin(w)\right)$
$= \frac{\sqrt{2}}{\sqrt{\pi}}i\frac{1}{w}\left(\cos(w) + \sin(w)\frac{1}{w}\right)$

$$\widehat{xf(x)} = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} x^2 e^{-iwx}\, dx = \frac{1}{\sqrt{2\pi}} \left(\left[\frac{x^2}{-iw} e^{-iwx}\right]_{-1}^{1} + \frac{2}{iw} \int_{-1}^{1} x e^{-iwx}\, dx \right)$$

$$= \frac{1}{\sqrt{2\pi}} \frac{-2}{-iw} i\sin(w) + \left(\sqrt{\frac{2}{\pi}}\, i\, \frac{1}{w}\left(\cos(w) - \frac{1}{w}\sin(w)\right)\right) \cdot \frac{2}{iw}$$

$$= \sqrt{\frac{2}{\pi}} \left(+\frac{1}{w}\sin(w) + \frac{2}{w^2}\cos(w) - \frac{2}{w^3}\sin(w)\right)$$

c) Plancherel : $\|f\|_2^2 = \|\hat{f}\|_2^2$

$$\cancel{t := \int |\widehat{xf(x)}|^2 = \frac{2}{\pi}\Big(\frac{1}{w^2}\sin^2(w) + \frac{2}{w^3}\cos\sin + \frac{2}{w^4}\sin^2(w) + \frac{2}{w^3}\cos\sin + \frac{4}{w^4}\cos^2}$$

$$\cancel{+ -\frac{4}{w^5}\cos\sin + \frac{4}{w^6}\sin^2(w) - \frac{2}{w^4}\sin^2 - \frac{4}{w^5}\sin\cos\Big)}$$

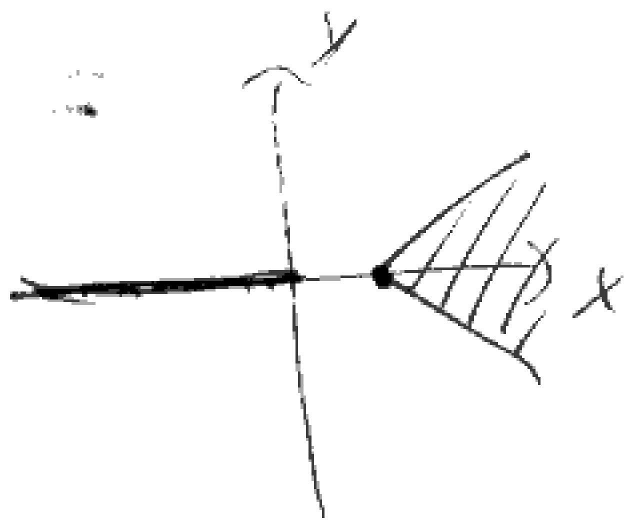
$$\cancel{= \frac{2}{\pi}\Big(\frac{1}{w^2}\sin^2 + \frac{4}{w^3}\cos\sin + \frac{4}{w^4}\cos^2 - \frac{4}{w^4}\sin^2}$$

$$\|\hat{f}(x)\|_2^2 = \frac{2}{\pi}\frac{1}{w^2}\Big(\cos^2 + \underbrace{\frac{2}{w}\cos\sin}_{\frac{\sin(2w)}{w}} + \frac{1}{w^2}\sin^2\Big)$$

$$\Rightarrow 1 = \frac{\pi}{2}\int_{\mathbb{R}} \|\hat{f}(x)\|^2 dx = \frac{\pi}{2}\int_{\mathbb{R}} \|f(x)\|^2 dx = \frac{\pi}{2}\int_{-1}^{1} x^2 dx = \frac{\pi}{3}.$$

Nr 5

a)



5a Beweis, 5b, 4
egal