Numerik 2 Blatt 6 Julian Bergmann

Aufgabe 1

$$y_{k+2} = y_{k+1} + h\left(\frac{5}{12} f_{k+2} + \frac{2}{3} f_{k+1} - \frac{1}{12} f_k\right)$$

$$\Rightarrow \quad y_{k+1} = y_k + h\left(\frac{5}{12} f_{k+1} + \frac{2}{3} f_k - \frac{1}{12} f_{k-1}\right)$$

$$y(x_{k+1}) = y(x_k) + h y'(x_k) + \frac{h^2}{2} y''(x_k) + \frac{h^3}{6} y'''(x_k) + O(h^4)$$

$$\begin{aligned} y_{k+1} &= y_k + h\Big(\frac{5}{12}\Big(y'(x_k) + h y''(x_k) + \frac{h^2}{2} y'''(x_k) + \frac{h^3}{6} y''''(x_k) + O(h^4)\Big) \\ &\qquad + \frac{2}{3} y'(x_k) - \frac{1}{12}\Big(y'(x_k) - h y''(x_k) + \frac{h^2}{2} y'''(x_k) - \frac{h^3}{6} y''''(x_k) + O(h^4)\Big)\Big) \\ &= y_k + h y'(x_k) + \frac{h^2}{2} y''(x_k) + \frac{h^3}{6} y'''(x_k) + \frac{h^4}{12} y''''(x_k) + O(h^5) \end{aligned}$$

$$y(x_{k+1}) = y_k + h y'(x_k) + \frac{h^2}{2} y''(x_k) + \frac{h^3}{6} y'''(x_k) + \frac{h^4}{24} y''''(x_k) + O(h^5)$$

$$\Rightarrow \quad y_{k+1} - y(x_{k+1}) = h^4 \underbrace{\left(\frac{1}{12} - \frac{1}{24}\right)}_{=C} y''''(x_k) + O(h^5) \quad \Rightarrow \quad C = \frac{1}{24}$$

$$\tilde{y}_{k+3} = \tilde{y}_{k+2} + h\left(\frac{23}{12} f_{k+2} - \frac{4}{3} f_{k+1} + \frac{5}{12} f_k\right)$$

$$\begin{aligned} \Rightarrow \quad \tilde{y}_{k+1} &= \tilde{y}_k + h\left(\frac{23}{12} f_k - \frac{4}{3} f_{k-1} + \frac{5}{12} f_{k-2}\right) \\ &= \tilde{y}_k + h\Big(\frac{23}{12} y'(x_k) - \frac{4}{3}\Big(y'(x_k) - h y''(x_k) + \frac{h^2}{2} y'''(x_k) - \frac{h^3}{6} y''''(x_k) \\ &\qquad + O(h^4)\Big) + \frac{5}{12}\Big(y'(x_k) - 2h y''(x_k) + 2h^2 y'''(x_k) \\ &\qquad - \frac{4}{3} h^3 y''''(x_k) + O(h^4)\Big)\Big) \\ &= \tilde{y}_k + h y'(x_k) + \frac{h^2}{2} y''(x_k) + \frac{h^3}{6} y'''(x_k) - \frac{h^4}{3} y''''(x_k) + O(h^5) \end{aligned}$$

$$\Rightarrow \quad \tilde{y}_{k+1} - y(x_{k+1}) = h^4 \underbrace{\left(-\frac{1}{3} - \frac{1}{24}\right)}_{=\tilde{C}} y''''(x_k) + O(h^5) \quad \Rightarrow \quad \tilde{C} = -\frac{3}{8}$$

$$\left|\frac{C}{C - \tilde{C}}\right| = \left|\frac{1/24}{1/24 + 3/8}\right| = \frac{1}{10}$$

$$\Rightarrow \quad |y_{k+1} - y(x_{k+1})| \approx \frac{1}{10} |y_{k+1} - \tilde{y}_{k+1}|$$

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Aufgabe 2

Trapez: $y_{k+1} = y_k + \frac{h}{2}(f_k + f_{k+1})$, $f_k = (1-2t_k)y(t_k)$

$y_0 = y(1) = 2$, $h_0 = 0{,}1$, $t_0 = 1$, $t_1 = 1{,}1$, $t_2 = 1{,}2$

$y_1 = y_0 + \frac{h_0}{2}((1-2t_0)y_0 + (1-t_1)y_1) =$

$= 2 + 0{,}05((1-2\cdot 1)\cdot 2 + (1-2(1{,}1))y_1) = 1{,}79245$

1. Schritt $y_2 = y_1 + \frac{h_0}{2}((1-2t_1)y_1 + (1-t_2)y_2)$

$= 1{,}79245 + 0{,}05((1-2\cdot 1{,}1)\cdot 1{,}79245 + (1-1{,}2)y_2)$

$= 1{,}57468$

2S-Adams-Bashforth: $\tilde{y}_{k+1} = y_k + \frac{h}{2}(3f_k - f_{k-1})$

$\tilde{y}_2 = y_1 + \frac{h_0}{2}(3(1-2t_1)y_1 - (1-2t_0)y_0)$

$= 1{,}79245 + 0{,}05(3\cdot(1-2\cdot 1{,}1)\cdot 1{,}79245 - (1-2\cdot 1)\cdot 2)$

$= 1{,}56981$

Nach Vorlesung: $|y_2 - y(x_2)| \approx \frac{1}{6}|y_2 - \tilde{y}_2|$!

$= 0{,}0008111$

$6\cdot 10^{-4} \le 0{,}0008 \le 4\cdot 10^{-3}$ $\Rightarrow h_3 = h_0 = 0{,}1$ $\Rightarrow t_3 = 1{,}3$

2. S. $y_3 = y_2 + \frac{h_0}{2}((1-2t_2)y_2 + (1-2t_3)y_3)$

$= 1{,}57468 + 0{,}05((1-2\cdot 1{,}2)\cdot 1{,}57468 + (1-2\cdot 1{,}3)y_3)$

$= 1{,}35597$

$\tilde{y}_3 = 1{,}35154$ $\Rightarrow |y_3 - y(x_3)| \approx \frac{1}{6}|y_3 - \tilde{y}_3| = 0.000738$

$6\cdot 10^{-4} \le 0{,}0007 \le 4\cdot 10^{-3}$ $\Rightarrow h_4 = h_0 = 0{,}1$ $\Rightarrow t_4 = 1{,}4$

3. $y_4 = 1{,}14449$ $\tilde{y}_4 = 1{,}14077$

$|y_4 - \tilde{y}_4|\cdot\frac{1}{6} = 0{,}00062$, $6\cdot 10^{-4} \le 0{,}00062 \le 4\cdot 10^{-3}$

$\Rightarrow h_5 = h_0 = 0{,}1$ $\Rightarrow t_5 = 1{,}5$

4. $y_5 = 0{,}946806$, $\tilde{y}_5 = 0{,}943356$, $|y_5 - \tilde{y}_5| = 0{,}000475$

$0{,}000475 < 6\cdot 10^{-4}$ $\Rightarrow h_6 = 0{,}2$, $t_6 = 1{,}7$

5. $y_6 = 0{,}610843$, $\tilde{y}_6 = 0{,}584731$, $|y_6 - \tilde{y}_6| = 0{,}004352$

$6\cdot 10^{-4} \le 0{,}0043 \not\le 4\cdot 10^{-3}$ $\Rightarrow h_7 = 0{,}1$, $t_7 = 1{,}8$

6. $y_7 = 0{,}4757$, $\tilde{y}_7 = 0{,}48562$, $|y_7 - \tilde{y}_7| = 0{,}0016532$

$6\cdot 10^{-4} \le 0{,}0017 \le 4\cdot 10^{-3}$ $\Rightarrow h_8 = h_7 = 0{,}1$, $t_8 = 1{,}9$

$y_8 = 0{,}363035$,

In[462]:= Clear[y, t, h, f, yb, abs, ys]

In[463]:= t[0] = 1; y[0] = 2; h[0] = 0.1; f[y_, t_] := (1 - 2 t) y; t[1] = 1.1; h[1] = 0.1;

$y[1] = y1 \,/.\, \text{Solve}\left[y1 == y[0] + \frac{h[0]}{2}\,(f[y[0], t[0]] + f[y1, t[1]]),\ y1\right][[1]];\ y[1]$

Out[464]= 1.79245

In[483]:=

$\text{For}\Big[i = 2,\ i < 9,\ i{+}{+},\ \Big($

$t[i] = t[i-1] + h[i-1];$

$y[i] = yi \,/.\, \text{Solve}\left[yi == y[i-1] + \frac{h[i-1]}{2}\,(f[yi, t[i]] + f[y[i-1], t[i-1]])\right][[1]];$

$yb[i] = y[i-1] + \frac{h[i-1]}{2}\,(3\,f[y[i-1], t[i-1]] - f[y[i-2], t[i-2]]);\ abs = \frac{1}{6}\,\text{Abs}[y[i] - yb[i]];$

$\text{If}\left[abs < 6 * 10^{-4},\ h[i] = 2\,h[i-1],\ \text{If}\left[abs > 4 * 10^{-3},\ h[i] = \frac{h[i-1]}{2},\ h[i] = h[i-1]\right]\right];$

ys[i] = {i, {"t" <> ToString[i], t[i]}, {"y" <> ToString[i], y[i]},
{"yb" <> ToString[i], yb[i]}, {"abs" <> ToString[i], abs}, {"h" <> ToString[i + 1], h[i]}};

$\Big)\Big]$

Table[ys[i], {i, 2, 8}] // TableForm

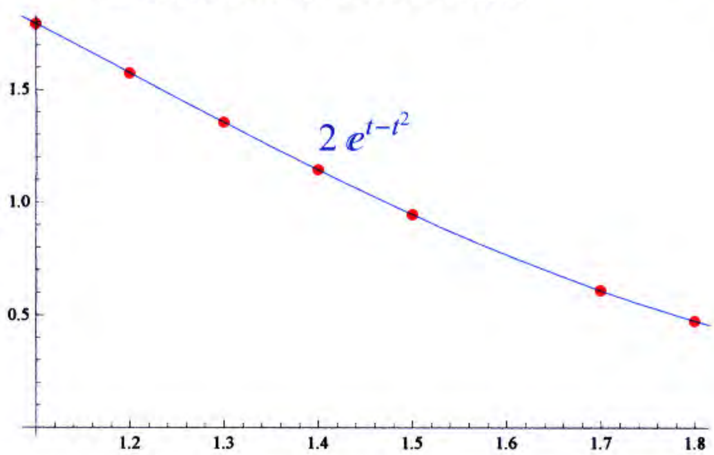
Out[484]//TableForm=

2	t2 1.2	y2 1.57468	yb2 1.56981	abs2 0.000811144	h3 0.1
3	t3 1.3	y3 1.35597	yb3 1.35154	abs3 0.000738324	h4 0.1
4	t4 1.4	y4 1.14449	yb4 1.14077	abs4 0.000620668	h5 0.1
5	t5 1.5	y5 0.946806	yb5 0.943956	abs5 0.000474986	h6 0.2
6	t6 1.7	y6 0.610843	yb6 0.584731	abs6 0.00435197	h7 0.1
7	t7 1.8	y7 0.4757	yb7 0.48562	abs7 0.00165324	h8 0.1
8	t8 1.9	y8 0.363035	yb8 0.363478	abs8 0.0000739726	h9 0.2

In[467]:=

```
In[468]:= Show[ListPlot[Table[{t[i], y[i]}, {i, 1, 7}], PlotMarkers → Automatic, PlotStyle -> Red],
  Plot[(y1[t1] /. DSolve[f[y1[t1], t1] == y1'[t1] && y1[1] == 2, y1[t1], t1][[1]]) /. t1 → x,
   {x, 1, 2}, PlotStyle → Blue], Epilog →
   {Inset[Style[(y1[t1] /. DSolve[f[y1[t1], t1] == y1'[t1] && y1[1] == 2, y1[t1], t1][[1]]) /. b → x /.
       t1 → t, 20, Blue], {1.45, 1.3}]}]
```

Out[468]=



Numerik 2 Blatt 6 Julian Bergmann

Aufgabe 3

2s-Adams-Bashforth:

$$y_{k+2}^{(0)} = y_{k+1}^{(c)} + \frac{h}{2}\left(3f(t_{k+1}, y_{k+1}^{(c)}) - f(t_k, y_k^{(c)})\right)$$

3s-Adams-Moulton:

$$y_{k+2}^{(l)} = y_{k+1}^{(c)} + h\left(\frac{5}{12} f(t_{k+2}, y_{k+2}^{(l-1)}) + \frac{2}{3} f(t_{k+1}, y_{k+1}^{(c)}) - \frac{1}{12} f(t_k, y_k^{(c)})\right)$$

AWP: $y' = f(t,y) = 2ty^2 e^{-t^2}$, $y(0)=1$, $t \in [0, 0{,}3]$, $h = 0{,}1$

$$y_1 = y_0 + h\left(f(x_0 + \tfrac{h}{2}, y_0 + \tfrac{h}{2} f(x_0, y_0))\right) = y_0 + h \cdot 2 \cdot (t_0 + \tfrac{h}{2}) e^{-(t_0+\frac{h}{2})} + \left(y_0 + \tfrac{h}{2} \cdot 2t_0\, y_0^2 e^{-t_0^2}\right)^2$$

$$= 1 + 0{,}1 \cdot 2 \cdot (0+0{,}05) e^{-0{,}05^2}(1 + 0{,}05 \cdot 2 \cdot 0 \cdot 1^2 e^{-0^2})^2 = 1{,}00998$$

Algorithmus:

$$y_{k+m}^{[0]} = \sum_{j=0}^{m-1} (h\beta_j^{(P)} f_{k+j} - \alpha_j^{(P)} y_{k+j})$$

Für $l = 1, \dots, N$:

$$f_{k+m}^{[l-1]} = f(x_{k+m}, y_{k+m}^{[l-1]})$$

$$y_{k+m}^{[l]} = h\beta_m^{(c)} f_{k+m}^{[l-1]} + \sum_{j=0}^{m-1} (h\beta_j^{(c)} f_{k+j} - \alpha_j^{(c)} y_{k+j})$$

Dann $y_{k+m} = y_{k+m}^{[N]}$ und $f_{k+m} = f_{k+m}^{[N]}$

$$y_2^{[0]} = h\left(\tfrac{3}{2} f(t_1, y_1) - \tfrac{1}{2} f(t_0, y_0)\right) + y_1$$
$$= 0{,}1 \cdot \left(\tfrac{3}{2} \cdot 2 \cdot 0{,}1 \cdot 1{,}00998^2 e^{-0{,}1^2} - \tfrac{1}{2} \cdot 2 \cdot 0\right) + 1{,}00998 = 1{,}04027$$

$$f_2^{[0]} = f(t_2, y_2^{[0]}) = 2 \cdot 0{,}2 \cdot 1{,}12405^2 \cdot e^{-0{,}2^2} = 0{,}415893$$

$$y_2^{[1]} = h\left(\tfrac{5}{12} f_2^{[0]}\right) + h\left(\tfrac{2}{3} f(t_1, y_1) - \tfrac{1}{12} f(t_0, y_0)\right) + y_1$$
$$= 1{,}04077$$

$$f_2^{[1]} = f(t_2, y_2^{[1]}) = 416291$$

$$y_2^{[2]} = h\left(\tfrac{5}{12} f_2^{[1]}\right) + h\left(\tfrac{2}{3} f(t_1, y_1) - \tfrac{1}{12} f(t_0, y_0)\right) + y_1 = 1{,}04079$$

$$f_2^{[2]} = f(t_2, y_2^{[2]}) = 0{,}416304 \qquad y_2^{[3]} = 1{,}04079 = y_2$$

$$y_3^{[0]} = h\left(\tfrac{3}{2} f(t_2, y_2) - \tfrac{1}{2} f(t_1, y_1)\right) + y_2 = 1{,}09313$$

$f_3^{[0]} = 0{,}655256$, $y_3^{[1]} = 1{,}09416$, $f_3^{[1]} = 0{,}656486$

$y_3^{[2]} = 1{,}09421$, $f_3^{[2]} = 0{,}656548$, $y_3^{[3]} = 1{,}09421$

$\Rightarrow y_3 = 1{,}09421$

$\Rightarrow$

```
Clear["Global`*"]
f[t_, y_] := 2 t y^2 E^(-t^2);
"k=0:";
y[0] = 1; h = 0.1; t[0] = 0; t[n_] := t[0] + h*n;
"k=1:";
y[1] = y[0] + h f[t[0] + h/2, y[0] + h/2 f[t[0], y[0]]]
```

$y(0) = 1 = e^0$

```
1.00998
```

$| y_1 - e^{t_1^2} | = | 1{,}00998 - e^{0{,}1^2} | = 0{,}000070167$

```
f[t_, y_] := 2 t y^2 E^(-t^2);
For[k = 2, k < 4, k++, (
  yl = h (3/2 f[t[k - 1], y[k - 1]] - 1/2 f[t[k - 2], y[k - 2]]) + y[k - 1]; Print[t[k], "→ y0=", yl];
  For[l = 1, l < 4, l++, (
    fl = 2 t[k] yl^2 E^(-t[k]^2);
    yl = h (5/12) fl + h (2/3 f[t[k - 1], y[k - 1]] - 1/12 f[t[k - 2], y[k - 2]]) + y[k - 1];
    Print[t[k], "→ fl=", fl, "\t y",
     k, "=", yl, "\t |y", k, "-e^(", t[k], "^2)|=", Abs[E^(t[k]^2) - yl]];
   )];
  y[
    k] =
   yl;)]
```

$\leftarrow f_{k+s} = f_{k+s}^{[l]}$ am Ende d. Schleife ⇒ betrachte letztes $y_k^{[l]}$ statt y_k!

$\leftarrow y_2^{[0]}$

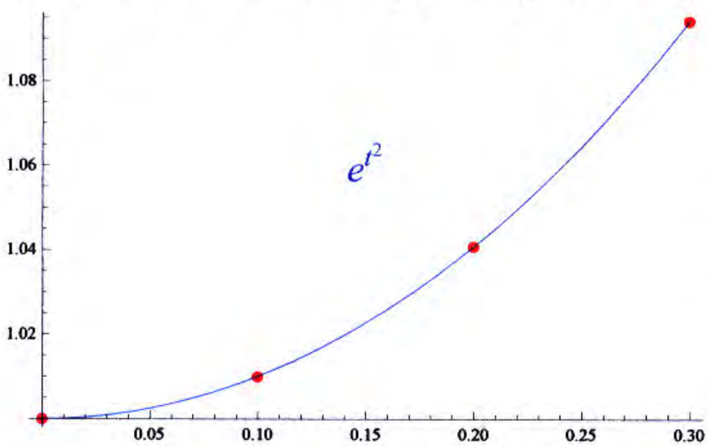
```
0.2→ y0=1.04027
0.2→ fl=0.415893     y2=1.04077     |y2-e^(0.2^2)|=0.0000415183
0.2→ fl=0.416291     y2=1.04079     |y2-e^(0.2^2)|=0.0000249486
0.2→ fl=0.416304     y2=1.04079     |y2-e^(0.2^2)|=0.0000243963
0.3→ y0=1.09313
0.3→ fl=0.655256     y3=1.09416     |y3-e^(0.3^2)|=0.0000150969
0.3→ fl=0.656486     y3=1.09421     |y3-e^(0.3^2)|=0.0000361828
0.3→ fl=0.656548     y3=1.09421     |y3-e^(0.3^2)|=0.0000387468
```

$y_2^{[1]} \rightarrow$ (second line), $y_3^{[0]} \rightarrow$ (fifth line)

$\Rightarrow = | y_2 - e^{t_2^2} |$

$\Rightarrow = | y_3 - e^{t_3^2} |$

```
Show[ListPlot[Table[{t[i], y[i]}, {i, 0, 3}], PlotStyle → Red, PlotMarkers → Automatic],
 Plot[E^(t^2), {t, 0, 0.3}, PlotStyle → Blue], Epilog → {Inset[Style["e^(t^2)", 20, Blue], {0.15, 1.06}]}]
```



⇒ Näherung ist der Funktion äußerst ähnlich!
Durch FP-Iteration wird der Fehler in $y_2^{[l]}$ geringer, in $y_3^{[l]}$ allerdings minimal größer.