Numerik 2 Blatt 10 Julian Bergmann

Aufgabe 1

a) $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$

$$(1-t)p_{n-1,0}(t) = (1-t)\binom{n-1}{0} t^0 (1-t)^{n-1}$$
$$= \binom{n-1}{0} t^0 (1-t)^n$$
$$= \frac{(n-1)!}{0!(n-1)!} \cdot 1 \cdot (1-t)^n$$
$$= (1-t)^n = p_{n,0}(t)$$

$$(1-t)p_{n-1,\nu}(t) + t p_{n-1,\nu-1}(t)$$
$$= (1-t)\binom{n-1}{\nu} t^\nu (1-t)^{n-1-\nu} + t\binom{n-1}{\nu-1} t^{\nu-1}(1-t)^{n-\nu}$$
$$= \frac{(n-1)!}{\nu!(n-1-\nu)!} t^\nu (1-t)^{n-\nu} + \frac{(n-1)!}{(\nu-1)!(n-\nu)!} t^\nu (1-t)^{n-\nu}$$
$$= \frac{(n-1)!(n-\nu) + (n-1)!\nu}{\nu!(n-\nu)!} t^\nu (1-t)^{n-\nu} = \frac{n!}{\nu!(n-\nu)!} t^\nu (1-t)^{n-\nu} = p_{n,\nu}(t)$$

$$t p_{n-1,n-1}(t) = t\binom{n-1}{n-1} t^{n-1}(1-t)^{n-1-n+1}$$
$$= t^n = \binom{n}{n} t^n (1-t)^{n-n} = p_{n,n}(t)$$

$\square$

b) $p_{4,0}(x) = \binom{4}{0} x^0 (1-x)^4 = (1-x)^4$

$p_{4,1}(x) = \binom{4}{1} x^1 (1-x)^3 = 4x(1-x)^3$

$p_{4,2}(x) = \binom{4}{2} x^2 (1-x)^2 = 6x^2(1-x)^2$

$p_{3,3}(x) = \binom{3}{3} x^3 (1-x)^0 = x^3$

$$p(t) = 2p_{4,0}(t) + 3p_{4,1}(t) + 4p_{4,2}(t) + 5t\,p_{3,3}(t)$$
$$= 2(1-t)^4 + 12t(1-t)^3 + 24t^2(1-t)^2 + 5t^4$$
$$= 2(1-4t+6t^2-4t^3+t^4) + 12t(1-3t+3t^2-t^3) + 24t^2(1-2t+t^2)$$
$$+ 5t^4$$
$$= 2 + 4t + 0t^2 - 20t^3 + 19t^4$$

c) i) $b^{(k)}(t) = \prod_{j=0}^{k-1}(n-j)\sum_{\mu=0}^{k}(-1)^\mu \binom{k}{\mu} b_{\nu-k+\mu,n-k}(t)$

$$b'_{n,\nu}(t) = (b_{n-1,\nu-1}(t) - b_{n-1,\nu}(t))\,n$$

$$Q'(t) = (p_{3,0}(t) + 3p_{3,1}(t) + 2p_{3,2}(t) + p_{3,3}(t))'$$
$$= 3(p_{2,-1}(t) - p_{2,0}(t)) + 9(p_{2,0}(t) - p_{2,1}(t)) + 6(p_{2,1}(t) - p_{2,2}(t))$$
$$+ 3t^2$$

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$$= 3(\ t)^2 + 9((1-t)^2 - 2(1-t)t) + 6(2t(1-t) - t^2) + 3t^2$$

(weiter auf Blatt 2/Seite 3)

$$= (\cancel{9+6t-3t^2}) + 9 - 18t + 9t^2 - 18t + 18t^2 + 12t - 12t^2 - 6t^2 + 3t^2$$
$$= 9t^2 - 18t + 6$$
$$\Rightarrow Q'(\tfrac{1}{4}) = \tfrac{33}{16}$$

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Aufgabe 1

c) i) $Q''(t) = 6\,p_{2,0}'(t) - 3\,p_{2,1}'(t) - 3\,p_{2,2}'(t)$

$= -12\,p_{1,0}(t) - 6(p_{1,0}(t) - p_{1,1}(t)) - 6\,p_{1,1}(t)$

$= -18\,p_{1,0}(t) = 18t - 18$

$\Rightarrow Q''(\frac{1}{4}) = -\frac{27}{2}$

ii) $Q(t) = \sum_{i=0}^{3} c_i\, p_{3,i}(t)$, $c_i = \{1, 3, 2, 1\}[i]$

$Q'(t) = \frac{3!}{2!} \sum_{k=0}^{2} \Delta c_k\, p_{2,k}(t)$

$= 3((c_1 - c_0)p_{2,0}(t) + (c_2 - c_1)p_{2,1}(t) + (c_3 - c_2)p_{2,2}(t))$

$= 6(1 - 2t + t^2) + (-3)(2t - 2t^2) + (-3)(t^2)$

$= 9t^2 - 18t + 6$

$\Rightarrow Q'(\frac{1}{4}) = \frac{33}{16}$, (wie (i))

$Q''(t) = \frac{3!}{1!} \sum_{k=0}^{1} \Delta^2 c_k\, p_{1,k}(t)$

$= 6((\Delta c_1 - \Delta c_0)p_{1,0}(t) + (\Delta c_2 - \Delta c_1)p_{1,1}(t))$

$= 6(((c_2 - c_1) - (c_1 - c_0))p_{1,0}(t) + ((c_3 - c_2) - (c_2 - c_1))p_{1,1}(t))$

$= -18\,p_{1,0}(t) = 18t - 8$

$\Rightarrow Q''(\frac{1}{4}) = -\frac{27}{2}$ (wie (i))

a) Algorithmus von de Casteljau

$p(t)=\sum_{k=0}^{n}\beta_k\, B_{n,k}(t)$, $n=17$, $\beta_k=\left(1-\frac{k}{17}\right)$

$$\sum_{v=0}^{16}\left(1-\frac{v}{17}\right)B_{17,v}\left(\tfrac{1}{2}\right)=\sum_{v=0}^{17}\left(1-\frac{v}{17}\right)B_{17,v}\left(\tfrac{1}{2}\right)-\underbrace{(1-1)B_{17,17}\left(\tfrac{1}{2}\right)}_{=0}$$

$\beta_0^{(0)}=1$ $\quad\beta_1^{(0)}=\frac{16}{17}\ \frac{15}{17}\ \frac{14}{17}\ \frac{13}{17}\ \frac{12}{17}\ \frac{11}{17}\ \frac{10}{17}\ \frac{9}{17}\ \frac{8}{17}\ \frac{7}{17}\ \frac{6}{17}\ \frac{5}{17}\ \frac{4}{17}\ \frac{3}{17}\ \frac{2}{17}\ \frac{1}{17}\ \frac{1}{17}$

$\cdot(1-t)=\frac{1}{2}$ $\quad + \quad$ $\cdot(t)=\frac{1}{2}$

$\beta_0^{(1)}$ $\frac{33}{34}\ \frac{31}{34}\ \frac{29}{34}\ \frac{27}{34}\ \frac{25}{34}\ \frac{23}{34}\ \frac{21}{34}\ \frac{19}{34}\ \frac{17}{34}\ \frac{15}{34}\ \frac{13}{34}\ \frac{11}{34}\ \frac{9}{34}\ \frac{7}{34}\ \frac{5}{34}\ \frac{3}{34}\ \frac{1}{34}$

$\beta_0^{(2)}=\frac{16}{17}\ \frac{15}{17}\ \frac{14}{17}\ \frac{13}{17}\ \frac{12}{17}\ \frac{11}{17}\ \frac{10}{17}\ \frac{9}{17}\ \frac{8}{17}\ \frac{7}{17}\ \frac{6}{17}\ \frac{5}{17}\ \frac{4}{17}\ \frac{3}{17}\ \frac{2}{17}\ \frac{1}{17}$

$\beta_0^{(3)}=\frac{31}{34}\ \frac{29}{34}\ \frac{27}{34}\ \frac{25}{34}\ \frac{23}{34}\ \frac{21}{34}\ \frac{19}{34}\ \frac{17}{34}\ \frac{15}{34}\ \frac{13}{34}\ \frac{11}{34}\ \frac{9}{34}\ \frac{7}{34}\ \frac{5}{34}\ \frac{3}{34}$

$\beta_0^{(4)}:\ \frac{15}{17}\ \frac{14}{17}\ \frac{13}{17}\ \frac{12}{17}\ \frac{11}{17}\ \frac{10}{17}\ \frac{9}{17}\ \frac{8}{17}\ \frac{7}{17}\ \frac{6}{17}\ \frac{5}{17}\ \frac{4}{17}\ \frac{2}{17}$

$\beta_0^{(5)}:\ \frac{29}{34}\ \frac{27}{34}\ \frac{25}{34}\ \frac{23}{34}\ \frac{21}{34}\ \frac{19}{34}\ \frac{17}{34}\ \frac{15}{34}\ \frac{13}{34}\ \frac{11}{34}\ \frac{9}{34}\ \frac{7}{34}\ \frac{5}{34}$

$\beta_0^{(6)}:\ \frac{14}{17}\ \frac{13}{17}\ \frac{12}{17}\ \frac{11}{17}\ \frac{10}{17}\ \frac{9}{17}\ \frac{8}{17}\ \frac{7}{17}\ \frac{6}{17}\ \frac{5}{17}\ \frac{4}{17}$

$\beta_0^{(7)}:\ \frac{27}{34}\ \frac{25}{34}\ \frac{23}{34}\ \frac{21}{34}\ \frac{19}{34}\ \frac{17}{34}\ \frac{15}{34}\ \frac{13}{34}\ \frac{11}{34}\ \frac{9}{34}$

$\beta_0^{(8)}:\ \frac{13}{17}\ \frac{12}{17}\ \frac{11}{17}\ \frac{10}{17}\ \frac{9}{17}\ \frac{8}{17}\ \frac{7}{17}\ \frac{6}{17}\ \frac{5}{17}$

$\beta_0^{(9)}=\frac{25}{34}\ \frac{23}{34}\ \frac{21}{34}\ \frac{19}{34}\ \frac{17}{34}\ \frac{15}{34}\ \frac{13}{34}\ \frac{11}{34}$

$\beta_0^{(10)}=\frac{12}{17}\ \frac{11}{17}\ \frac{10}{17}\ \frac{9}{17}\ \frac{8}{17}\ \frac{7}{17}\ \frac{6}{17}$

$\beta_0^{(11)}:\ \frac{23}{34}\ \frac{21}{34}\ \frac{19}{34}\ \frac{17}{34}\ \frac{15}{34}\ \frac{13}{34}$

$\beta_0^{(12)}:\ \frac{11}{17}\ \frac{10}{17}\ \frac{9}{17}\ \frac{8}{17}\ \frac{7}{17}$

$\beta_0^{(13)}=\frac{21}{34}\ \frac{19}{34}\ \frac{17}{34}\ \frac{15}{34}$

$\frac{21}{34}\ \frac{19}{34}\ \frac{17}{34}\ \frac{15}{34}\ \frac{13}{34}=\beta_4^{(13)}$

$\frac{10}{17}\ \frac{9}{17}\ \frac{8}{17}\ \frac{7}{17}=\beta_3^{(14)}$

$\frac{19}{34}\ \frac{17}{34}\ \frac{15}{34}=\beta_2^{(15)}$

$\frac{9}{17}\ \frac{8}{17}=\beta_1^{(16)}$

$\frac{17}{34}=\frac{1}{2}=\beta_0^{(17)}$

$\Rightarrow\ \sum_{v=0}^{16}\left(1-\frac{v}{17}\right)B_{17,v}\left(\tfrac{1}{2}\right)=\frac{1}{2}$

(2/3)

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Aufgabe 2

a) $P(x) = 5x^4 - 3x^3 + x^2 + 5x - 2$

Bernstein-Polynome $(n=4)$:

$p_{4,0}(x) = (1-x)^4 \quad = 1 - 4x + 6x^2 - 4x^3 + x^4$

$p_{4,1}(x) = (1-x)^3 \cdot 4x \quad = 4x - 12x^2 + 12x^3 - 4x^4$

$p_{4,2}(x) = (1-x)^2 \cdot 6x^2 \quad = 6x^2 - 12x^3 + 6x^4$

$p_{4,3}(x) = (1-x) \cdot 4x^3 \quad = 4x^3 - 4x^4$

$p_{4,4}(x) = x^4 \quad = x^4$

Koeffizientenvergleich: (Ansatz: $P(x) = \sum_{i=0}^{4} \beta_i \, p_{4,i}(x)$) Konstanten

$$\left|\begin{array}{l} 5 = \beta_0 - 4\beta_1 + 6\beta_2 - 4\beta_3 + \beta_4 \\ -3 = -4\beta_0 + 12\beta_1 - 12\beta_2 + 4\beta_3 \\ 1 = -12\beta_1 + 6\beta_0 + 6\beta_2 \\ 5 = -4\beta_0 + 4\beta_1 \\ -2 = \beta_0 \end{array}\right| \Leftrightarrow \left|\begin{array}{l} 6 = \beta_4 \\ 3/2 = \beta_3 \\ 2/3 = \beta_2 \\ -3/4 = \beta_1 \\ -2 = \beta_0 \end{array}\right|$$

$\Rightarrow P(x) = -2\, p_{4,0}(x) - \frac{3}{4} p_{4,1}(x) + \frac{2}{3} p_{4,2}(x) + \frac{3}{2} p_{4,3}(x) + 6\, p_{4,4}(x)$

b) $P(t) = \sum_{k=0}^{n} \beta_k \, p_{n,k}(t)$

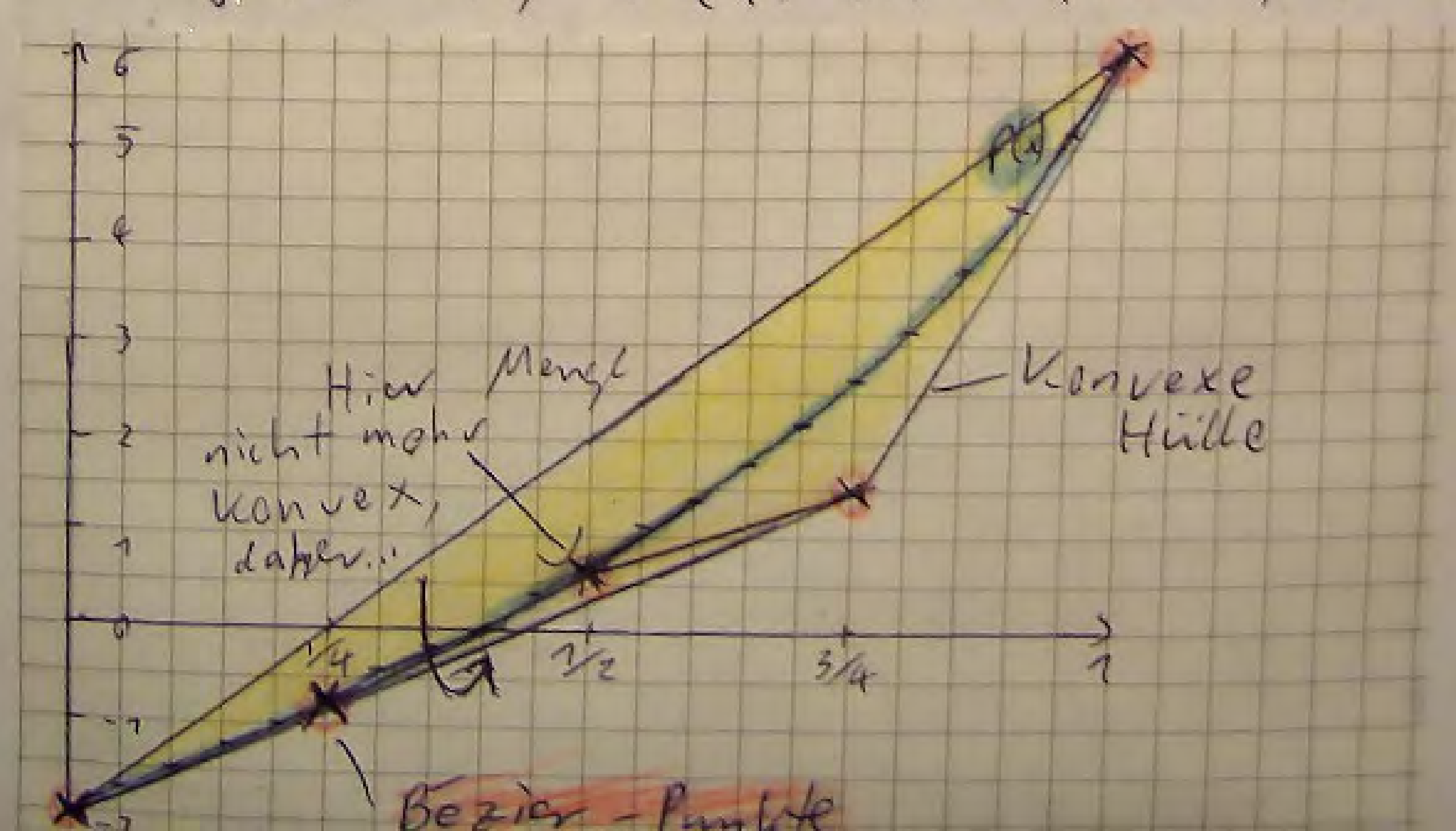
Bézier-Punkte

$\Rightarrow \beta_0 = -2,\ \beta_1 = -\frac{3}{4},\ \beta_2 = \frac{2}{3},\ \beta_3 = \frac{3}{2},\ \beta_4 = 6$ (siehe a))

$\Rightarrow$ Bezier-Koeffizienten

$\Rightarrow$ Bezier-Punkte: $\left(\frac{i}{n}, \beta_i\right) \in \mathbb{R}^2$

$\Rightarrow P_0 = (0, -2),\ P_1 = \left(\frac{1}{4}, -\frac{3}{4}\right),\ P_2 = \left(\frac{1}{2}, \frac{2}{3}\right),\ P_3 = \left(\frac{3}{4}, \frac{3}{2}\right),\ P_4 = (1, 6)$



c) de Casteljau: $p(t) = \sum_{k=0}^{n} p_k B_{n,k}(t)$

$t = \frac{1}{2}$ $\qquad\qquad\qquad\qquad$ $t = \frac{1}{4}$

$$
\begin{array}{llllll}
\Rightarrow \beta_i^{(0)}: & -2 & -\frac{3}{4} & \frac{2}{3} & \frac{7}{2} & 6 \\
 & \cdot(1-t) & \cdot t & & & \\
\beta_i^{(1)}: & -\frac{11}{8} & -\frac{1}{24} & \frac{13}{12} & \frac{15}{4} & \\
\beta_i^{(2)}: & -\frac{17}{24} & \frac{25}{48} & \frac{29}{12} & & \\
\beta_i^{(3)}: & -\frac{3}{32} & \frac{47}{32} & & & \\
\beta_i^{(4)}: & \frac{11}{16} = \beta_0^{(4)} = p(\frac{1}{2}) & & & &
\end{array}
\qquad
\begin{array}{lllll}
-2 & -\frac{3}{4} & \frac{2}{3} & \frac{7}{2} & 6 \\
-\frac{27}{16} & -\frac{19}{48} & \frac{7}{8} & \frac{27}{8} & \\
-\frac{131}{96} & -\frac{5}{64} & \frac{27}{16} & & \\
-\frac{267}{256} & \frac{69}{256} & & & \\
-\frac{183}{256} = \beta_0^{(4)} = p(\frac{1}{4}) & & & &
\end{array}
$$

d)
$$p'(t) = \frac{4!}{3!}\sum_{k=0}^{3} \Delta c_k B_{3,k}(t) \quad , \; c_k \text{ entsprechen den } \beta_i \text{ aus (a)-(c)!}$$
$$= 4\big((c_1-c_0)B_{3,0}(t) + (c_2-c_1)B_{3,1}(t) + (c_3-c_2)B_{3,2}(t) + (c_4-c_3)B_{3,3}(t)\big)$$
$$= 4\big(\tfrac{5}{4}B_{3,0}(t) + \tfrac{17}{12}B_{3,1}(t) + \tfrac{5}{6}B_{3,2}(t) + \tfrac{9}{2}B_{3,3}(t)\big)$$
$$= 5B_{3,0}(t) + \tfrac{17}{3}B_{3,1}(t) + \tfrac{10}{3}B_{3,2}(t) + 18B_{3,3}(t)$$
$$p''(t) = \frac{4!}{2!}\sum_{k=0}^{2} \Delta^2 c_k B_{2,k}(t)$$
$$= 12\big((\Delta c_1 - \Delta c_0)B_{2,0}(t) + (\Delta c_2 - \Delta c_1)B_{2,1}(t) + (\Delta c_3 - \Delta c_2)B_{2,2}(t)\big)$$
$$= 12\big(((c_2-c_1)-(c_1-c_0))B_{2,0}(t) + ((c_3-c_2)-(c_2-c_1))B_{2,1}(t) + ((c_4-c_3)-(c_3-c_2))B_{2,2}(t)\big)$$
$$= 2B_{2,0}(t) + (-7)B_{2,1}(t) + 44B_{2,2}(t)$$
$$p''(-\tfrac{1}{2}) = 2\binom{2}{0}(1+\tfrac{1}{2})^2 - 7\binom{2}{1}\cdot(-\tfrac{1}{2})\cdot(1+\tfrac{1}{2}) + 44\binom{2}{2}(-\tfrac{1}{2})^2 = 26$$
$$p''(\tfrac{1}{2}) = 2\binom{2}{0}(1-\tfrac{1}{2})^2 - 7\binom{2}{1}\cdot\tfrac{1}{2}\cdot(1-\tfrac{1}{2}) + 44\binom{2}{2}(\tfrac{1}{2})^2 = 8$$

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Aufgabe 3

a) $t \in [0,1]: s(t) := p_{3,0}(t) + p_{3,3}(t) = (1-t)^3 + t^3 = 1 - 3t + 3t^2$

$t \in [1,2]: s(t) := p_{3,0}(t-1) + 2p_{3,1}(t-1) + 4p_{3,2}(t-1) + 3p_{3,3}(t-1)$
$= (2-t)^3 + 6(t-1)(2-t)^2 + 12(t-1)^2(2-t) + 3(t-1)^3$
$= 5 - 15t + 15t^2 - 4t^3$

$t \in [2,3]: s(t) := 3p_{3,0}(t-2) + 2p_{3,1}(t-2) - 2p_{3,2}(t-2)$
$= 3(3-t)^3 + 6(t-2)(3-t)^2 - 6(t-2)^2(3-t)$
$= -99 + 141t - 63t^2 + 9t^3$

$t \in [3,4]: s(t) := 2p_{3,1}(t-3) + 8p_{3,2}(t-3) + 8p_{3,3}(t-3)$
$= 6(t-3)(4-t)^2 + 24(t-3)^2(4-t) + 8(t-3)^3$
$= 360 - 336t + 102t^2 - 10t^3$

$t \in [0,1]: s'(t) = 6t - 3$
$t \in [1,2]: s'(t) = 30t - 15 - 12t^2$
$t \in [2,3]: s'(t) = 141 - 126t + 27t^2$
$t \in [3,4]: s'(t) = 204t - 336 - 30t^2$

$t = 1: \quad s'(1)|_{t\in[0,1]} = 3 = s'(1)|_{t\in[1,2]}$

$t = 2: \quad s'(2)|_{t\in[1,2]} = 60 - 15 - 48 = -3 = 141 - 252 + 108 = s'(2)|_{t\in[2,3]}$

$t = 3: \quad s'(3)|_{t\in[2,3]} = 141 - 378 + 243 = 6 = 612 - 336 - 270 = s'(3)|_{t\in[3,4]}$

$\Rightarrow$ s' stetig an „Sprungstellen" (Definitionsänderungen)
sonst auch stetig, da Polynome. $\Rightarrow$ $s(t)$ stetig d.bar!

b) $t \in [0,1]: s''(t) = 6$ $\quad s''(1) = 6$
$t \in [1,2]: s''(t) = 30 - 24t$ $\quad s''(1) = 6 \quad s''(2) = -18$
$t \in [2,3]: s''(t) = 54t - 126$ $\quad s''(2) = -18 \quad s''(3) = 36$
$t \in [3,4]: s''(t) = 204 - 60t$ $\quad s''(3) = 24$

Alles gilt weiterhin: $s''(1)|_{t\in[0,1]} = s''(1)|_{t\in[1,2]}, = 6$

$s''(2)_{t\in[1,2]} = s''(2)|_{t\in[2,3]}, = -18$

aber: $s''(3)_{t\in[2,3]} = 36 \neq 24 = s''(3)|_{t\in[3,4]}$

$\Rightarrow$ $s(t)$ nur zweimal st. d.bar auf dem Intervall $[0,3]$