H9-1

a) $\vec{F}(\vec{r}) = (x + 2y + az \;,\; bx - 3y - z \;,\; 4x + cy + 2z)$

(Kraft $\vec{F}$ konservativ $\Longleftrightarrow$) $\vec{\nabla} \times \vec{F} = \vec{0}$ $\quad [\Rightarrow \exists \phi \mid \vec{F} = -\vec{\nabla}\phi]$

$$\vec{\nabla} \times \vec{F} = \begin{pmatrix} \partial_y F_z - \partial_z F_y \\ \partial_z F_x - \partial_x F_z \\ \partial_x F_y - \partial_y F_x \end{pmatrix} = \begin{pmatrix} c + 1 \\ a - 4 \\ b - 2 \end{pmatrix} \stackrel{!}{=} \vec{0}$$

$\Rightarrow$ $\vec{F}$ konservativ für $a = 4$, $b = 2$, $c = -1$.

$\Rightarrow$ $\vec{F}(\vec{r}) = (x + 2y + 4z \;,\; 2x - 3y - z \;,\; 4x - y + 2z)$

b) $\rightsquigarrow$ Bestimmung des Potentials: $\quad \vec{F} = -\vec{\nabla}\phi$

„Sukzessive Antiintegration"

$F_x(\vec{r}) = -\frac{d}{dx}\phi(\vec{r}) \quad \Rightarrow \quad \phi(\vec{r}) = -\int dx \; F_x$

Integrationskonstante, die von y & z abhängt

$$\Rightarrow \phi(\vec{r}) = -\int dx \; (x + 2y + az) = -\frac{1}{2}x^2 - 2xy - 4xz + g_1(y,z)$$

$$-\frac{d}{dy}\phi(\vec{r}) = 2x - \frac{d}{dy}g_1(y,z) \stackrel{!}{=} 2x - 3y - z = F_y(\vec{r})$$

$$\Rightarrow \frac{d}{dy}g_1(y,z) = 3y + z$$

Integrationskonstante, die nur von z abhängt

$$\Rightarrow g_1(y,z) = \int dy \; (3y + z) = \frac{3}{2}y^2 + zy + g_2(z)$$

H9-2

$$\longrightarrow \quad \phi(\vec{r}) = -\frac{1}{2}x^2 - 2xy - 4xz + \frac{3}{2}y^2 + yz + g_2(z)$$

$$\longrightarrow -\frac{d}{dz}\phi(\vec{r}) = -\left(-4x + y + \frac{d}{dz}g_2(z)\right) \overset{!}{=} 4x - y + 2z = F_z(\vec{r})$$

$$\longrightarrow \quad \frac{d}{dz}g_2(z) = -2z$$

"wirkliche" Integrationskonstante

$$\longrightarrow \quad g_2(z) = \int dz\, -2z = -z^2 + g_3$$

$$\longrightarrow \quad \phi(\vec{r}) = -\frac{1}{2}x^2 - 2xy - 4xz + \frac{3}{2}y^2 + yz - z^2 + g_3$$

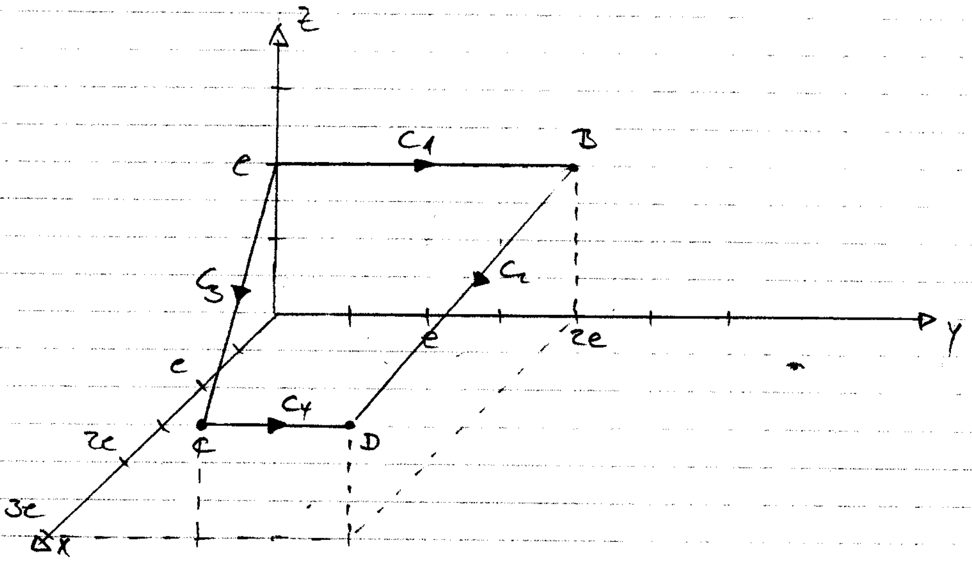
Kann man natürlich auch "raten" und dann durch Ableiten verifizieren.

H 10-1

$$\vec{F} = -\gamma \frac{mM}{r^2} \vec{e_r}$$

$$A = (0,0,e) \xrightarrow{C_1} B = (0, 2e, e) \xrightarrow{C_2} D = (3e, 2e, e)$$

$$A = (0,0,e) \xrightarrow{C_3} C = (3e, e, e) \xrightarrow{C_4} D = (3e, 2e, e)$$



$$\int_C d\vec{r}\, \vec{F}(\vec{r}) = \int dt\, \frac{d\vec{r}}{dt} \cdot \vec{F}(\vec{r}(t)) = \int dt \cdot \frac{d\vec{r}}{dt} \cdot \underbrace{(-\gamma m M)}_{=:\sigma} \frac{\vec{r}}{r^3}$$

wobei $\vec{r}(t)$ eine Parameterdarstellung des Weges C ist

- Linienintegral C_1 : $A \to B$

$$\vec{r}(t) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ 0 \\ e \end{pmatrix}, \ t: 0 \to 2e \quad \Rightarrow \quad \frac{d\vec{r}}{dt} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad |\vec{r}(t)| = \left(t^2 + e^2\right)^{1/2}$$

$$\Rightarrow \int_{C_1} d\vec{r}\, \vec{F} = \sigma \int_0^{2e} dt \cdot \frac{1}{\sqrt{t^2+e^2}^{\,3}} \cdot \begin{array}{c} \frac{d\vec{r}}{dt} \\ \downarrow \\ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \end{array} \cdot \begin{array}{c} \vec{r}(t) \\ \downarrow \\ \begin{pmatrix} 0 \\ t \\ e \end{pmatrix} \end{array}$$

H10-2

$$\Rightarrow \int_{C_1} d\vec{r}\,\vec{F} = \delta \int_0^{2e} dt \, \frac{1}{\sqrt{t^2+e^2}^3} \cdot t$$

$u = t^2 + e^2$, $\frac{du}{dt} = 2t$

$$\int dt \frac{t}{\sqrt{t^2+e^2}^3} = \int du \, \frac{1}{2} u^{-3/2} = -u^{-1/2} \frac{1}{\ } = -\frac{1}{\sqrt{t^2+e^2}}$$

für später

$$\Rightarrow \int_{C_1} d\vec{r}\,\vec{F} = -\frac{\delta}{\sqrt{t^2+e^2}}\Bigg|_0^{2e} = \delta\left(-\frac{1}{\sqrt{5e^2}} + \frac{1}{\sqrt{e^2}}\right) = \delta \frac{1}{e}\left(1 - \frac{1}{\sqrt{5}}\right)$$

- Linienintegral C_2: $B \to D$

$$\vec{r}(t) = \begin{pmatrix} t \\ 2e \\ e \end{pmatrix}, \quad t: 0 \to 3e \qquad \Rightarrow \frac{d\vec{r}}{dt} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$|\vec{r}(t)| = \sqrt{t^2 + 5e^2}$$

$$\Rightarrow \int_{C_2} d\vec{r}\,\vec{F} = \delta \int_0^{3e} dt \, \frac{1}{\sqrt{t^2+5e^2}^3} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} t \\ 2e \\ e \end{pmatrix} = \delta \int_0^{3e} dt \, \frac{t}{\sqrt{t^2+5e^2}^3}$$

$$= -\frac{\delta}{\sqrt{t^2+5e^2}}\Bigg|_0^{3e} = \delta\left(-\frac{1}{\sqrt{14e^2}} + \frac{1}{\sqrt{5e^2}}\right) = \delta \frac{1}{e}\left(\frac{1}{\sqrt{5}} - \frac{1}{\sqrt{14}}\right)$$

H10-3

- Linienintegral C_3 : $A \to C$

$$\vec{r}(t) = \begin{pmatrix} 3t \\ t \\ e \end{pmatrix}, \quad t: 0 \to e \qquad \frac{d\vec{r}}{dt} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$$

$$|\vec{r}(t)| = \sqrt{10t^2 + e^2}$$

$$\Rightarrow \int_{C_3} d\vec{r}\, \vec{F}(\vec{r}) = \delta \int_0^e dt \frac{1}{\sqrt{10t^2+e^2}^3} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 3t \\ t \\ e \end{pmatrix} = \delta \int_0^e dt \frac{10t}{\sqrt{10t^2+e^2}^3}$$

$$= \delta\left(-\frac{\cancel{10}}{\cancel{10}} \cdot \frac{1}{\sqrt{10t^2+e^2}}\right)\Bigg|_0^e = \delta\left(\frac{1}{\sqrt{e^2}} - \frac{1}{\sqrt{11e^2}}\right) = \delta \frac{1}{e}\left(1 - \frac{1}{\sqrt{11}}\right)$$

- Linienintegral C_4 : $C \to D$

$$\vec{r}(t) = \begin{pmatrix} 3e \\ t \\ e \end{pmatrix}, \quad t: e \to 2e, \quad \frac{d\vec{r}}{dt} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |\vec{r}(t)| = \sqrt{t^2 + 10e^2}$$

$$\to \int_{C_4} d\vec{r}\, \vec{F}(\vec{r}) = \delta \int_e^{2e} dt \frac{1}{\sqrt{t^2+10e^2}^3} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 3e \\ t \\ e \end{pmatrix} = \delta \int_e^{2e} dt \frac{t}{\sqrt{t^2+10e^2}^3}$$

$$= \delta\left(-\frac{1}{\sqrt{t^2+10e^2}}\right)\Bigg|_e^{2e} = \left(\frac{1}{\sqrt{11e^2}} - \frac{1}{\sqrt{14e^2}}\right)\delta = \delta\frac{1}{e}\left(\frac{1}{\sqrt{11}} - \frac{1}{\sqrt{14}}\right)$$

$$\to \quad A \to B \to D = \frac{\delta}{e}\left(1 - \frac{1}{\sqrt{5}}\right) + \frac{\delta}{e}\left(\frac{1}{\sqrt{5}} - \frac{1}{\sqrt{14}}\right) = \left(1 - \frac{1}{\sqrt{14}}\right)$$

$$A \to C \to D = \frac{\delta}{e}\left(1 - \frac{1}{\sqrt{11}}\right) + \frac{\delta}{e}\left(\frac{1}{\sqrt{11}} - \frac{1}{\sqrt{14}}\right) = \left(1 - \frac{1}{\sqrt{14}}\right)$$

Gesamtlinienintegral ist wegunabhängig, da Gravitationskraft $\vec{F}$ konservativ, bzw. existiert das zugehörige Potential.