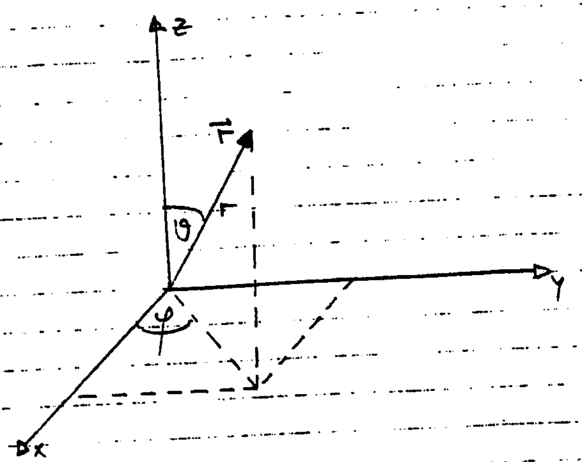
[redacted]

H21

H21-1



Kugelkoordinaten (r, ϑ, φ)

$$x = r \sin\vartheta \cos\varphi$$
$$y = r \sin\vartheta \sin\varphi \quad \Longrightarrow$$
$$z = r \cos\vartheta$$

$$r = \sqrt{x^2+y^2+z^2}$$
$$\varphi = \arctan\left(\frac{y}{x}\right)$$
$$\vartheta = \arccos\left(\frac{z}{r}\right) = \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right)$$

(a) Umrechnung von kartesischen Koordinaten auf Kugelkoordinaten

$$\vec{a}_1 = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} \Rightarrow r = \sqrt{1^2+(2)^2+4^2} = \sqrt{21}$$
$$\varphi = \arctan(-2)$$
$$\vartheta = \arctan\left(\frac{\sqrt{5}}{4}\right)$$

H21-2

(b) Einheitsvektoren der Kugelkoordinaten

$$\vec{e}_k = \frac{\frac{\partial \vec{r}}{\partial k}}{\left|\frac{\partial \vec{r}}{\partial k}\right|}$$

• $$\frac{\partial \vec{r}}{\partial r} = \frac{\partial}{\partial r}\begin{pmatrix} r \sin\vartheta \cos\varphi \\ r \sin\vartheta \sin\varphi \\ r \cos\vartheta \end{pmatrix} = \begin{pmatrix} \sin\vartheta \cos\varphi \\ \sin\vartheta \sin\varphi \\ \cos\vartheta \end{pmatrix}$$

$$\Rightarrow \left|\frac{\partial \vec{r}}{\partial r}\right| = \left[\sin^2\vartheta \cos^2\varphi + \sin^2\vartheta \sin^2\varphi + \cos^2\vartheta\right]^{1/2}$$

$$= \left[\sin^2\vartheta\,(\underbrace{\cos^2\varphi + \sin^2\varphi}_{1}) + \cos^2\vartheta\right]^{1/2}$$

$$= \left[\sin^2\vartheta + \cos^2\vartheta\right]^{1/2}$$

$$= 1$$

$$\Rightarrow \vec{e}_r = \frac{\frac{\partial \vec{r}}{\partial r}}{\left|\frac{\partial \vec{r}}{\partial r}\right|} = \begin{pmatrix} \sin\vartheta \cos\varphi \\ \sin\vartheta \sin\varphi \\ \cos\vartheta \end{pmatrix}$$

• $$\frac{\partial \vec{r}}{\partial \vartheta} = \begin{pmatrix} r \cos\vartheta \cos\varphi \\ r \cos\vartheta \sin\varphi \\ -r \sin\vartheta \end{pmatrix}$$

$$\Rightarrow \left|\frac{\partial \vec{r}}{\partial \vartheta}\right| = \left[r^2 \cos^2\vartheta \cos^2\varphi + r^2 \cos^2\vartheta \sin^2\varphi + r^2 \sin^2\vartheta\right]^{1/2}$$

$$= r\left[\cos^2\vartheta\,(\cos^2\varphi + \sin^2\varphi) + \sin^2\vartheta\right]^{1/2}$$

$$= r\left[\cos^2\vartheta + \sin^2\vartheta\right]^{1/2}$$

$$= r$$

$$\Rightarrow \vec{e}_\vartheta = \frac{\frac{\partial \vec{r}}{\partial \vartheta}}{\left|\frac{\partial \vec{r}}{\partial \vartheta}\right|} = \frac{1}{r}\begin{pmatrix} r \cos\vartheta \cos\varphi \\ r \cos\vartheta \sin\varphi \\ -r \sin\vartheta \end{pmatrix} = \begin{pmatrix} \cos\vartheta \cos\varphi \\ \cos\vartheta \sin\varphi \\ -\sin\vartheta \end{pmatrix}$$

H21-3

- $$\frac{d\vec r}{d\varphi} = \begin{pmatrix} -r\sin\vartheta\sin\varphi \\ r\sin\vartheta\cos\varphi \\ 0 \end{pmatrix}$$

$$\Rightarrow \left|\frac{d\vec r}{d\varphi}\right| = \left[r^2\sin^2\vartheta\sin^2\varphi + r^2\sin^2\vartheta\cos^2\varphi\right]^{1/2}$$

$$= r\cdot\sin\vartheta$$

$$\Rightarrow \vec e_\varphi = \frac{\frac{d\vec r}{d\varphi}}{\left|\frac{d\vec r}{d\varphi}\right|} = \frac{1}{r\sin\vartheta}\begin{pmatrix} -r\sin\vartheta\sin\varphi \\ r\sin\vartheta\cos\varphi \\ 0 \end{pmatrix} = \underline{\underline{\begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}}}$$

Aufgabe nicht gestellt

Zu zeigen: $\vec e_\vartheta, \vec e_\varphi$ bilden Orthogonales Netz

$$\vec e_\vartheta\cdot\vec e_\varphi = \begin{pmatrix} \cos\vartheta\cos\varphi \\ \cos\vartheta\sin\varphi \\ -\sin\vartheta \end{pmatrix}\cdot\begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}$$

$$= -\cos\vartheta\cos\varphi\sin\varphi + \cos\vartheta\sin\varphi\cos\varphi = 0 \checkmark$$

ebenfalls: $\vec e_r\cdot\vec e_\vartheta = 0$, $\vec e_r\cdot\vec e_\varphi = 0$ (ohne Beweis)

c) Zu zeigen: $\vec e_r, \vec e_\vartheta, \vec e_\varphi$ bilden Rechtssystem

$$\vec e_r \times \vec e_\vartheta = \begin{pmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix} \times \begin{pmatrix} \cos\vartheta\cos\varphi \\ \cos\vartheta\sin\varphi \\ -\sin\vartheta \end{pmatrix}$$

$$= \begin{pmatrix} -\sin^2\vartheta\cdot\sin\varphi - \cos^2\vartheta\sin\varphi \\ \cos^2\vartheta\cos\varphi - (-\sin^2\vartheta\cos\varphi) \\ \sin\vartheta\cos\vartheta\cancel{\cos\varphi\sin\varphi} - \sin\vartheta\cancel{\cos\vartheta\sin\varphi}\cos\varphi \end{pmatrix}$$

$$= \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} = \vec e_\varphi$$

gilt zyklisch: $\Rightarrow$ $\vec e_\vartheta\times\vec e_\varphi = \vec e_r$, $\vec e_\varphi\times\vec e_r = \vec e_\vartheta$

H22 -1

Gegeben: $\vec{g}(\vec{r}) = f(r)\,\vec{e}_r$ und $\vec{h}(\vec{r}) = f(\vec{r})\,\vec{e}_\varphi$

Ferner gilt: $\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}(\sin\vartheta\, A_\vartheta) + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\varphi}A_\varphi$.

und

$$\vec{\nabla}\times\vec{A} = \frac{1}{r^2\sin\vartheta}\begin{vmatrix} \vec{e}_r & r\,\vec{e}_\vartheta & r\sin\vartheta\,\vec{e}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\vartheta} & \frac{\partial}{\partial\varphi} \\ A_r & r A_\vartheta & r\sin\vartheta\, A_\varphi \end{vmatrix}$$

Feld $\vec{g}(\vec{r}) = f(r)\vec{e}_r$:

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Mit:

$$\vec\nabla\cdot\vec g(\vec r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 f(r)\right) = \frac{\partial f}{\partial r} + \frac{2f}{r} \qquad \text{und}$$

$$\vec\nabla\times\vec g(\vec r) = \frac{1}{r^2\sin\vartheta}\begin{vmatrix} \vec e_r & r\vec e_\vartheta & r\sin\vartheta\,\vec e_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\vartheta} & \frac{\partial}{\partial\varphi} \\ f(r) & 0 & 0 \end{vmatrix} = 0$$

folgt für die gesuchten 2-ten Ableitungen:

$$\underline{\vec\nabla(\vec\nabla\cdot\vec g)} = \vec e_r\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial r}+\frac{2f}{r}\right) = \underline{\vec e_r\left[\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial f}{\partial r}\right)-\frac{2f}{r^2}\right]}$$

$$\underline{\vec\nabla\times(\underbrace{\vec\nabla\times\vec g}_{=0}) = 0}$$

Feld $\vec h(\vec r) = f(\vec r)\,\vec e_\varphi$:

Mit: $\vec\nabla\cdot\vec h(\vec r) = \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\varphi}\left(f(r)\right) = 0$ und

$$\vec\nabla\times\vec h(\vec r) = \frac{1}{r^2\sin\vartheta}\begin{vmatrix} \vec e_r & r\vec e_\vartheta & r\sin\vartheta\,\vec e_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\vartheta} & \frac{\partial}{\partial\varphi} \\ 0 & 0 & r\sin\vartheta f(r) \end{vmatrix} = \frac{1}{r}\cot\vartheta\, f(r)\,\vec e_r - \left(\frac{\partial f}{\partial r}+\frac{f}{r}\right)\vec e_r$$

folgt für die gesuchten 2-ten Ableitungen:

$$\underline{\vec\nabla(\vec\nabla\cdot\vec h) = 0}$$

H22-3

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{h}) = \vec{\nabla} \times \left(\frac{1}{r} \cot\vartheta \, f \, \vec{e}_r - \left(\frac{\partial f}{\partial r} + \frac{f}{r} \right) \vec{e}_\vartheta \right)$$

$$= \frac{1}{r^2 \sin\vartheta} \begin{vmatrix} \vec{e}_r & r\vec{e}_\vartheta & r\sin\vartheta \, \vec{e}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \vartheta} & \frac{\partial}{\partial \varphi} \\ \frac{1}{r}\cot\vartheta \, f & -r\frac{\partial f}{\partial r} - f & 0 \end{vmatrix} = \vec{e}_\varphi \left[\frac{f}{r^2 \sin^2\vartheta} - \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial f}{\partial r} \right]$$

Selected Solutions to Problems

Teile (a)-(d): P29

C.1 Chapter 2 Problems

P29

Problem 2-1

(a) In the spaceship frame, events 1 and 2 do not occur at the same space point, that is, event 2 occurs on Earth. However, both events 1 and 2 occur at the same place in the Earth frame, so it is a proper time interval in the Earth frame.

(b) Following the same reasoning as in part (a), the time interval between events 2 and 3 is not a proper time interval in either frame.

(c) The time interval between events 1 and 3 is a proper time interval in the spaceship frame, but not in the Earth frame.

(d) Because the time between events 1 and 2 is proper time interval in the Earth frame, all that the spaceship sees is a dilated time value,

$$t_2' = \gamma t_e = \frac{10}{\sqrt{1-\frac{v^2}{c^2}}} \min = 12.5 \min.$$

(e) The velocity of Earth according to the spaceship is $0.6c$. Also, the time between events 2 and 1, according to the spaceship is 12.5 minutes, as found in part (d). So the distance of Earth at event 2, according to the spaceship is

$$l_2' = 12.5 \cdot 60 \cdot 0.6c = 1.35 \cdot 10^{11} \text{ m}.$$

Teile (e)-(h): H23

(f) The time between events 3 and 2 is (according to the spaceship)

$$t_3' - t_2' = \frac{l_2'}{c} = 7.5 \min.$$

And we know the time of event 2 according to the spaceship. So the time of event 3 is

$$t_3' = 7.5 + 12.5 = 20 \min.$$

(g) From Earth's perspective, when Earth emits the pulse (event 2), the spaceship is at a distance

$$l_2 = 10 \min \cdot 0.6c = 1.08 \cdot 10^{11} \text{ m}.$$

When the pulse reaches the spaceship, the spaceship has moved an additional distance. Let the time for the pulse to travel to the spaceship be called Δt, where

$$\Delta t = t_3 - t_2$$

and

$$c\Delta t = 1.08 \cdot 10^{11} + v\Delta t$$

$$\Delta t = \frac{1.08 \cdot 10^{11}}{2.9979 \cdot 10^8 \cdot 0.4} = 900\,\text{s} = 15 \min.$$

So the time of event 3 according to Earth is

$$t_3 = t_2 + \Delta t = 25 \min.$$

(h) We know that the time interval between events 3 and 1 is a proper time in the spaceship frame (part c). So the time interval between events 3 and 1 in the Earth frame should just be the dilated value of the time interval in the rocket frame:

$$t_3 - t_1 = \gamma(t_3' - t_1').$$

Now $t_1 = t_1' = 0$, so we should have

$$t_3 = \gamma t_3'.$$

Let us see if this is true: $t_3 = 25$ minutes, while $t_3' = 20$ minutes.

$$t_3 = \gamma t_3' = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} 20 = \frac{5}{4} \cdot 20 = 25 \min.$$

Hence our results are consistent.