1 Präsenzaufgabe 18.10.12

Erinnerung

Kommutator:

$$[A+B,C+D] = [A+B,C] + [A+B,D] = [A,C] + [B,C]$$

Operatoren:

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger)\ , \qquad p = -i\sqrt{\frac{\hbar m\omega}{2}}(a-a^\dagger)\ , \qquad <a|b> = \delta_{ab}$$

a) $[x,p]\psi(t) = (xp - px)\psi(t) = -xi\hbar\frac{\partial}{\partial x}\psi(t) + i\hbar\frac{\partial x}{\partial x}\psi(t) = i\hbar\psi(t)$

$$\begin{aligned}
&[a,a^\dagger]\psi(t) = aa^\dagger\psi(t) - a^\dagger a\psi(t)\\
&= \tfrac{1}{2\hbar m\omega}(m\omega x + ip)(m\omega x - ip)\psi(t) - \tfrac{1}{2\hbar m\omega}(m\omega x - ip)(m\omega x + ip)\psi(t)\\
&= \tfrac{1}{2\hbar m\omega}(((m\omega x)^2 - m\omega xip + ipm\omega x + pp)\psi(t)\\
&\quad - ((m\omega x)^2 + m\omega xip - ipm\omega x + pp)\psi(t))\\
&= \tfrac{1}{2\hbar m\omega}(-2m\omega xip + 2ipm\omega x)\psi(t) = \psi(t)
\end{aligned}$$

b)
$$\begin{aligned}
&(aa^\dagger + a^\dagger a)\psi(t)\\
&= \tfrac{1}{2\hbar m\omega}(((m\omega x)^2 - m\omega xip + ipm\omega x + pp)\psi(t)\\
&\quad + ((m\omega x)^2 + m\omega xip - ipm\omega x + pp)\psi(t))\\
&= \tfrac{1}{\hbar m\omega}(((m\omega x)^2 + p^2)\psi(t))\\
&= \tfrac{1}{2\hbar m\omega}(((m\omega x)^2 - m\omega xip + ipm\omega x + pp)\psi(t)\\
&\quad + ((m\omega x)^2 + m\omega xip - ipm\omega x + pp)\psi(t))\\
&H = \tfrac{p^2}{2m} + \tfrac{m}{2}\omega^2x^2 = \tfrac{1}{2m}\tfrac{\hbar m\omega}{\hbar m\omega}p^2 + \tfrac{\hbar m\omega}{\hbar m\omega}\tfrac{1}{2m}m^2\omega^2x^2\\
&\Rightarrow H\psi(t) = \tfrac{1}{2}\omega\hbar(aa^\dagger + a^\dagger a)
\end{aligned}$$

c) Ansatz: $\frac{a^{\dagger m}|0>}{\sqrt{m!}}$

$$\begin{aligned}
&m = 0: && \frac{a^{\dagger 0}|0>}{\sqrt{0!}} = 1|0>\\
&m = m+1: && \frac{a^{\dagger m+1}|0>}{\sqrt{(m+1)!}} = \frac{\sqrt{(n+1)!}}{\sqrt{(n+1)!}}|m+1>
\end{aligned}$$

d)
$$\begin{aligned}
&H|n> = \tfrac{1}{2}\omega\hbar(aa^\dagger + a^\dagger a)|n> = \tfrac{1}{2}\omega\hbar(a\sqrt{n+1}|n+1> + a^\dagger\sqrt{n}|n-1>)\\
&= \tfrac{1}{2}\omega\hbar(\sqrt{n+1}^2|n> + \sqrt{n}^2|n>)\\
&= \underbrace{\frac{1}{2}\omega\hbar(2n+1)}_{EnergieEW}|n>
\end{aligned}$$

e)
$$\begin{aligned}
&(\Delta x)^2 = <x^2> - <x>^2 = <0|\tfrac{\hbar}{2m\omega}(a+a^\dagger)^2|0> - (<0|\sqrt{\tfrac{\hbar}{2m\omega}}(a+a^\dagger)|0>)^2\\
&= \tfrac{\hbar}{2m\omega} <0|\sqrt{1}^2|0> = \tfrac{\hbar}{2m\omega}
\end{aligned}$$