Numerik 1 Blatt 2 Julian Bergmann

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Aufgabe 1

a) $A = \begin{pmatrix} -1 & 3 & 2 \\ -3 & 1 & 0 \\ -5 & -2 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} -5 & -2 & -3 \\ -3 & 1 & 0 \\ -1 & 3 & 2 \end{pmatrix} \Rightarrow L_{12} = \frac{3}{5}, \quad L_{13} = \frac{1}{5}$

$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ +\frac{3}{5} & 1 & 0 \\ +\frac{1}{5} & 0 & 1 \end{pmatrix}\begin{pmatrix} -5 & -2 & -3 \\ 0 & \frac{11}{5} & \frac{9}{5} \\ 0 & \frac{17}{5} & \frac{13}{5} \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ +\frac{3}{5} & 1 & 0 \\ +\frac{1}{5} & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} -5 & -2 & -3 \\ 0 & \frac{17}{5} & \frac{13}{5} \\ 0 & \frac{11}{5} & \frac{9}{5} \end{pmatrix} \Rightarrow L_{23} = \frac{11}{17}$

$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ +\frac{3}{5} & 1 & 0 \\ +\frac{1}{5} & 0 & 1 \end{pmatrix}\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{\text{nach vorne holen vertauscht Zeilen}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & +\frac{11}{17} & 1 \end{pmatrix}\begin{pmatrix} -5 & -2 & -3 \\ 0 & \frac{17}{5} & \frac{13}{5} \\ 0 & 0 & \frac{2}{17} \end{pmatrix}$

$\Rightarrow L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{5} & 1 & 0 \\ \frac{3}{5} & \frac{11}{17} & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$L = G_1^{-1} G_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{5} & 1 & 0 \\ \frac{3}{5} & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{11}{17} & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{5} & 1 & 0 \\ -\frac{3}{5} & 0 & 1 \end{pmatrix}}_{G_1}{}^{-1}\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{11}{17} & 1 \end{pmatrix}}_{G_2}{}^{-1}$

b) $LRx_i = b_i, \quad b_1 = (1,0,0)^T, \; b_2 = (0,1,0)^T, \; b_3 = (0,0,1)^T$

$\Rightarrow Lc_i = b_i \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ \frac{1}{5} & 1 & 0 & 0 \\ \frac{3}{5} & \frac{11}{17} & 1 & 0 \end{array}\right)_{b_1} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -\frac{1}{5} \\ 0 & \frac{11}{17} & 1 & -\frac{3}{5} \end{array}\right) \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -\frac{1}{5} \\ 0 & 0 & 1 & -\frac{8}{17} \end{array}\right)_{c_1}$

$\Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ \frac{1}{5} & 1 & 0 & 1 \\ \frac{3}{5} & \frac{11}{17} & 1 & 0 \end{array}\right)_{b_2} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & \frac{11}{17} & 1 & 0 \end{array}\right) \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -\frac{11}{17} \end{array}\right)_{c_2}$

$\Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ \frac{1}{5} & 1 & 0 & 0 \\ \frac{3}{5} & \frac{11}{17} & 1 & 1 \end{array}\right)_{b_3} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)_{c_3}$

$\Rightarrow Rx_i = c_i \Rightarrow \left(\begin{array}{ccc|c} -5 & -2 & -3 & 1 \\ 0 & \frac{17}{5} & \frac{13}{5} & -\frac{1}{5} \\ 0 & 0 & \frac{2}{17} & -\frac{8}{17} \end{array}\right)_{c_1} \Leftrightarrow \left(\begin{array}{ccc|c} -5 & -2 & -3 & 1 \\ 0 & \frac{17}{5} & \frac{13}{5} & -\frac{1}{5} \\ 0 & 0 & 1 & -4 \end{array}\right)$

$\Leftrightarrow \left(\begin{array}{ccc|c} -5 & -2 & 0 & -11 \\ 0 & \frac{17}{5} & 0 & \frac{51}{5} \\ 0 & 0 & 1 & -4 \end{array}\right) \Leftrightarrow \left(\begin{array}{ccc|c} -5 & 0 & 0 & -5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -4 \end{array}\right) \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -4 \end{array}\right)_{x_1}$

$\Rightarrow \left(\begin{array}{ccc|c} -5 & -2 & -3 & 0 \\ 0 & \frac{17}{5} & \frac{13}{5} & 1 \\ 0 & 0 & \frac{2}{17} & -\frac{11}{17} \end{array}\right) \Leftrightarrow \left(\begin{array}{ccc|c} -5 & -2 & 0 & -\frac{33}{2} \\ 0 & \frac{17}{5} & 0 & \frac{153}{10} \\ 0 & 0 & 1 & -\frac{11}{2} \end{array}\right) \Leftrightarrow \left(\begin{array}{ccc|c} -5 & 0 & 0 & -\frac{15}{2} \\ 0 & 1 & 0 & \frac{9}{2} \\ 0 & 0 & 1 & -\frac{11}{2} \end{array}\right)$

$\Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{3}{2} \\ 0 & 1 & 0 & \frac{9}{2} \\ 0 & 0 & 1 & -\frac{11}{2} \end{array}\right)_{x_2}$

$\Rightarrow \left(\begin{array}{ccc|c} -5 & -2 & -3 & 0 \\ 0 & \frac{17}{5} & \frac{13}{5} & 0 \\ 0 & 0 & \frac{2}{17} & 1 \end{array}\right) \Leftrightarrow \left(\begin{array}{ccc|c} -5 & -2 & 0 & \frac{51}{2} \\ 0 & \frac{17}{5} & 0 & -\frac{221}{10} \\ 0 & 0 & 1 & \frac{17}{2} \end{array}\right) \Leftrightarrow \left(\begin{array}{ccc|c} -5 & 0 & 0 & \frac{25}{2} \\ 0 & 1 & 0 & -\frac{13}{2} \\ 0 & 0 & 1 & \frac{17}{2} \end{array}\right)$

sorry

$\Rightarrow$

$$\Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & -5/2 \\ 0 & 1 & 0 & -13/2 \\ 0 & 0 & 1 & \underbrace{17/2}_{x_3} \end{array}\right)$$

$$\Rightarrow (PA)^{-1} = \begin{pmatrix} \overbrace{1}^{x_1} & \overbrace{3/2}^{x_2} & \overbrace{-5/2}^{x_3} \\ 3 & 9/2 & -13/2 \\ -4 & -11/2 & 17/2 \end{pmatrix} = A^{-1}P^{-1}$$

$$\Rightarrow (PA)^{-1}P = A^{-1} = \begin{pmatrix} 3/2 & -5/2 & 1 \\ 9/2 & -13/2 & 3 \\ -11/2 & 17/2 & 4 \end{pmatrix}$$

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Aufgabe 2

1. $\operatorname{cond}(AB) = \|AB\|\cdot\|(AB)^{-1}\| \leq \|A\|\cdot\|B\|\|B^{-1}\|\|A^{-1}\|$

$= \|A\|\|A^{-1}\|\cdot\|B\|\|B^{-1}\| = \operatorname{cond}(A)\operatorname{cond}(B)$ ✓

2. $\operatorname{cond}(\alpha A) = \|\alpha A\|\cdot\|(\alpha A)^{-1}\| = \|\alpha A\|\cdot\|A^{-1}\alpha^{-1}\|$

$= |\alpha|\cdot\|A\|\cdot\|A^{-1}\|\cdot|\frac{1}{\alpha}| = |\frac{\alpha}{\alpha}|\,\|A\|\|A^{-1}\| = \operatorname{cond}(A)$ ✓

3. $\operatorname{cond}_2(U) = \|U\mathbb{1}\|_2\cdot\|(U\mathbb{1})^{-1}\|_2 = \|U\mathbb{1}\|_2\cdot\|\mathbb{1}U'\|_2$

-1 ?

$= \|\mathbb{1}\|_2\|\mathbb{1}\|_2 = 1$ $\left(\begin{array}{l} \max_{\|x\|_2=1}\|Qx\|_2 = \max_{\|x_2\|_2=1}\|x_2\|_2 = 1 \\ \max_{\|x\|_2=1}\|Q^{-1}x\|_2 = \quad \text{''} \quad = 1 \end{array}\right)$

↑ orth. Matrizen sind längentreu

4. $\operatorname{cond}_2(UA) = \|UA\|_2\cdot\|(UA)^{-1}\|_2$

$\|UA\|_2 = \max_{\|x\|_2=1}\|UAx\|_2 = \max_{\|x\|_2=1}\|Ax\|_2 = \|A\|_2$

$\|A^{-1}\|_2 = \|A^{-1}U^{-1}U\|_2 \leq \|A^{-1}U^{-1}\|_2\|U\|_2 = \|(UA)^{-1}\|_2 \leq \|A^{-1}\|_2\|U^{-1}\|_2 = \|A^{-1}\|_2$

$\Rightarrow \|A^{-1}\|_2 = \|(UA)^{-1}\|_2$

$\Rightarrow \|UA\|_2\cdot\|(UA)^{-1}\|_2 = \|A\|_2\|A^{-1}\|_2 = \operatorname{cond}_2(A)$ ✓

5. $\operatorname{cond}_2(A) = \|A\|_2\|A^{-1}\|_2$

$\|Ax\|_2 = \left(\sum_{j=1}^n\left(|\sum_{k=1}^n a_{jk}x_k|\right)^2\right)^{1/2} \underset{C.S.}{\leq} \left(\sum_{j=1}^n \underbrace{\left(\sum_{k=1}^n|a_{jk}|^2\right)}_{\text{Frobenius}}\underbrace{\left(\sum_{k=1}^n|x_k|^2\right)}_{\text{2-Vektor-Norm}}\right)^{1/2} = \|A\|_F\|x\|$ ✓

~~$\Rightarrow \operatorname{cond}_2(A) = \|A\|_2\|A^{-1}\|_2 \leq \|A\|_F\|A^{-1}\|_F = \operatorname{cond}_F(A)$~~

~~$\|A\|_F = \left(\sum_{j=1}^n\left(\sum_{k=1}^n|a_{jk}|^2\right)\right)^{1/2} \leq \frac{1}{\sqrt{n}}\sum_{j=1}^n\sum_{k=1}^n|a_{jk}| \leq \frac{n}{\sqrt{n}}\cdot\max_{1\leq j\leq n}\sum_{k=1}^n|a_{jk}| = \sqrt{n}\|A\|$~~

~~n. singul. ⇒ rang = dim.~~

~~$\Rightarrow \operatorname{cond}_F(A) = \|A\|_F\|A^{-1}\|_F \leq \sqrt{n}\|A\|_\infty\sqrt{n}\|A^{-1}\|_\infty = n\operatorname{cond}_\infty(A)$~~

$\|A\|_F = \sqrt{\sum_{j=1}^n\sum_{k=1}^n|a_{jk}|^2} \leq \sqrt{n\max_{1\leq j\leq 1}\sum_{k=1}^n|a_{jk}|^2} \neq \sqrt{n}\,\|A\|_\infty$ ~~(A)~~ ✓

$\Rightarrow \operatorname{cond}_2(A) = \|A\|_F\|A^{-1}\|_F \leq \sqrt{n}\,\|A\|_\infty\sqrt{n}\|A^{-1}\|_\infty = n\operatorname{cond}_\infty(A)$