Numerik 1 Blatt 3 Julian Bergmann

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Nr. 1

LR-Zerlegung:

$$A = \begin{pmatrix} 25 & 5 & -5 & 0 \\ 5 & 10 & 5 & -6 \\ -5 & 5 & 9 & 2 \\ 0 & -6 & 2 & 22 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{5} & 1 & 0 & 0 \\ -\frac{1}{5} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 25 & 5 & -5 & 0 \\ 0 & 9 & 6 & -6 \\ 0 & 6 & 8 & 2 \\ 0 & -6 & 2 & 22 \end{pmatrix} \quad (l_{12} = \tfrac{1}{5},\ l_{13} = -\tfrac{1}{5},\ l_{14} = 0)$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{5} & 1 & 0 & 0 \\ -\frac{1}{5} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{2}{3} & 1 & 0 \\ 0 & -\frac{2}{3} & 0 & 1 \end{pmatrix} \begin{pmatrix} 25 & 5 & -5 & 0 \\ 0 & 9 & 6 & -6 \\ 0 & 0 & 4 & 6 \\ 0 & 0 & 6 & 18 \end{pmatrix} \quad (l_{23} = \tfrac{2}{3},\ l_{24} = -\tfrac{2}{3})$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{5} & 1 & 0 & 0 \\ -\frac{1}{5} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{2}{3} & 1 & 0 \\ 0 & -\frac{2}{3} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{3}{2} & 1 \end{pmatrix} \begin{pmatrix} 25 & 5 & -5 & 0 \\ 0 & 9 & 6 & -6 \\ 0 & 0 & 4 & 6 \\ 0 & 0 & 0 & 9 \end{pmatrix} \quad (l_{34} = \tfrac{3}{2})$$

✓

$$\Rightarrow L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{5} & 1 & 0 & 0 \\ -\frac{1}{5} & \frac{2}{3} & 1 & 0 \\ 0 & -\frac{2}{3} & \frac{3}{2} & 1 \end{pmatrix}$$ ✓

$\tilde{L}\tilde{L}^T$-Zerlegung:

1. A symmetrisch : $a_{ij} = a_{ji} \Rightarrow$ erfüllt
2. A pos. definit : EW pos u. Reell

$$0 \stackrel{!}{=} \det(A - \mathbb{1}\lambda) = -6\big((25-\lambda)\cdot 5\cdot 2 + (-5)(-6)\cdot 5 - (25-\lambda)(9-\lambda)(-6) - 5(-5)\cdot 2\big)$$
$$-2\big((25-\lambda)(10-\lambda)\cdot 2 + 5\cdot(-6)(-5) - (25-\lambda)\cdot 5\cdot(-6) - 5\cdot 5\cdot 2\big)$$
$$+22\big((25-\lambda)(10-\lambda)(9-\lambda) + 5\cdot 5\cdot(-5) + 5\cdot 5\cdot(-5)$$
$$-(25-\lambda)\cdot 5\cdot 5 - (-5)(10-\lambda)(-5) - (9-\lambda)\cdot 5\cdot 5\big)$$
$$= \lambda^4 - 66\lambda^3 + 1418\lambda^2 - 10196\lambda + 8100$$

$\Rightarrow \lambda_1 \approx 0{,}9;\ \lambda_2 \approx 13{,}7;\ \lambda_3 \approx 23{,}5;\ \lambda_4 \approx 27{,}9 \Rightarrow$ erfüllt ✓

$\tilde{l}_{11}^2 = a_{11} = 25 \Rightarrow \tilde{l}_{11} = 5$ | $\tilde{l}_{21}\tilde{l}_{11} = a_{21} = 5 \Rightarrow \tilde{l}_{21} = 1$

$\tilde{l}_{31}\tilde{l}_{11} = a_{31} = -5 \Rightarrow \tilde{l}_{31} = -1$ | $\tilde{l}_{21}^2 + \tilde{l}_{22}^2 = a_{22} = 10 \Rightarrow \tilde{l}_{22} = 3$

$\tilde{l}_{31}\tilde{l}_{21} + \tilde{l}_{32}\tilde{l}_{22} = a_{32} = 5 \Rightarrow \tilde{l}_{32} = 2$ | $\tilde{l}_{41}\tilde{l}_{11} = a_{41} = 0 \Rightarrow \tilde{l}_{41} = 0$

$\tilde{l}_{41}\tilde{l}_{21} + \tilde{l}_{42}\tilde{l}_{22} = a_{42} = -6 \Rightarrow \tilde{l}_{42} = -2$ | $\tilde{l}_{32}^2 + \tilde{l}_{31}^2 + \tilde{l}_{33}^2 = a_{33} = 9 \Rightarrow \tilde{l}_{33} = 2$

$\tilde{l}_{41}\tilde{l}_{31} + \tilde{l}_{42}\tilde{l}_{32} + \tilde{l}_{43}\tilde{l}_{33} = a_{43} = 2 \Rightarrow \tilde{l}_{43} = 3$ | $\tilde{l}_{41}^2 + \tilde{l}_{42}^2 + \tilde{l}_{43}^2 + \tilde{l}_{44}^2 = a_{44} = 22 \Rightarrow \tilde{l}_{44} = 3$

$$\Rightarrow \tilde{L} = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ -1 & 2 & 2 & 0 \\ 0 & -2 & 3 & 3 \end{pmatrix}$$ ✓

Vergleich:

Spalten von $\tilde{L}$ dividiert durch jeweiliges Diagonalelement ergibt L.

Zeilen von $\tilde{L}^T$ multipliziert mit jew. Diagonalelement ergibt R.

✓

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Nr. 2

a) Lösbarkeit:

$$\det(A) = \begin{vmatrix} 0 & 1 & 1 \\ 0{,}5 & 1{,}0001 & 3 \\ 1 & 2 & 4 \end{vmatrix} = 3-2-1{,}0001 = 0{,}9999 \neq 0 \Rightarrow \text{lösbar}$$

Lösen:

$$\left(\begin{array}{ccc|c} 0 & 1 & 1 & 2 \\ 0{,}5 & 1{,}0001 & 3 & 3 \\ 1 & 2 & 4 & 4 \end{array}\right) \overset{piv\, a_{33}^{(1)}}{\Leftrightarrow} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \left(\begin{array}{ccc|c} 4 & 2 & 1 & 4 \\ 3 & 1{,}0001 & 0{,}5 & 3 \\ 1 & 1 & 0 & 2 \end{array}\right) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad l_{21} = \tfrac{3}{4},\; l_{31} = \tfrac{1}{4}$$

$$\Leftrightarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 3/4 & 1 & 0 \\ 1/4 & 0 & 1 \end{pmatrix} \left(\begin{array}{ccc|c} 4 & 2 & 1 & 4 \\ 0 & -0{,}4999 & -\frac{1}{4} & 0 \\ 0 & 0{,}5 & -\frac{1}{4} & 1 \end{array}\right) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad piv.: a_{32}^{(2)} = 0{,}5,\; l_{32} = -0{,}9998$$

$$\Leftrightarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 3/4 & 1 & 0 \\ 1/4 & 0 & 1 \end{pmatrix}}_{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -0{,}9998 & 1 \end{pmatrix}} \left(\begin{array}{ccc|c} 4 & 2 & 1 & 4 \\ 0 & 0{,}5 & -\frac{1}{4} & 1 \\ 0 & 0 & -0{,}49995 & 0{,}9998 \end{array}\right) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ 3/4 & -0{,}9998 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 4 & 2 & 1 \\ 0 & 0{,}5 & -\frac{1}{4} \\ 0 & 0 & -0{,}49995 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad b = (4,\ 1,\ 0{,}9998)^T$$

$$Ax = P_1 L R P_2 x = b \Rightarrow Lc = b:$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 1/4 & 1 & 0 & 1 \\ 3/4 & -0{,}9998 & 1 & 0{,}9998 \end{array}\right) \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2{,}0002 \end{array}\right) \Rightarrow c = (4,\ 0,\ -2{,}0002)^T$$

$$R P_2 x = c:$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & 4 \\ -1/4 & 0{,}5 & 0 & 0 \\ -0{,}49995 & 0 & 0 & -2{,}0002 \end{array}\right) \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 4 & 4 \\ -1/4 & 0{,}5 & 0 & 0 \\ 1 & 0 & 0 & 4{,}0008 \end{array}\right) \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 4 & 4 \\ 0 & 1 & 0 & 2{,}0004 \\ 1 & 0 & 0 & 4{,}0008 \end{array}\right)$$

$$\Leftrightarrow \left(\begin{array}{ccc|c} 0 & 0 & 1 & -1{,}0004 \\ 0 & 1 & 0 & 2{,}0004 \\ 1 & 0 & 0 & 4{,}0008 \end{array}\right) \Rightarrow x = (4{,}0008;\ 2{,}0004;\ -1{,}0004)^T$$

⇒ Rückseite

b) $\text{cond}_\infty(A) = \|A^{-1}\| \cdot \|A\|$

$$A^{-1} \Rightarrow \left(\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 0{,}5 & 1{,}0001 & 3 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right) \Leftrightarrow \left(\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0{,}0001 & 1 & 0 & 1 & -0{,}5 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right)$$

$$\Leftrightarrow \left(\begin{array}{ccc|ccc} 0 & 0{,}9999 & 0 & 1 & -1 & 0{,}5 \\ 0 & 0{,}0001 & 1 & 0 & 1 & -0{,}5 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right) \Leftrightarrow \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 1{,}0001 & -1{,}0001 & 0{,}50005 \\ 0 & 0 & 1 & -0{,}0001 & 1{,}0001 & -0{,}50005 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right)$$

$$\Leftrightarrow \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 1{,}0001 & -1{,}0001 & 0{,}50005 \\ 0 & 0 & 1 & -0{,}0001 & 1{,}0001 & -0{,}50005 \\ 1 & 0 & 0 & -1{,}9998 & -2{,}0002 & 2{,}0001 \end{array}\right)$$

$$\Rightarrow A^{-1} = \begin{pmatrix} -1{,}9998 & -2{,}0002 & 2{,}0001 \\ 1{,}0001 & -1{,}0001 & 0{,}50005 \\ -0{,}0001 & 1{,}0001 & -0{,}50005 \end{pmatrix}$$

⇒

$\Rightarrow \|A\|_\infty = 7 \quad , \quad \|A^{-1}\|_\infty = 6{,}0001$

$\Rightarrow \text{cond}_\infty(A) = 7 \cdot 6{,}0001 = 42{,}0007$ ✓

$$\frac{\|\delta x\|}{\|x\|} \le \frac{\text{cond}_\infty(A)}{1-\text{cond}_\infty(A)\frac{\|\delta A\|}{\|A\|}}\left(\frac{\|\delta A\|}{\|A\|}+\frac{\|\delta b\|}{\|b\|}\right) = \frac{42{,}0007}{1-42{,}0007\cdot 10^{-5}}\cdot 2\cdot 10^{-5}$$

$\uparrow 10^{-5}$, $\nearrow 10^{-5}$ ← 10^{-8}

(−1)

$= 8{,}404 \cdot 10^{-4}$

Nachtrag zu 2a):

$Lc = bP_1^{-1}$: Bereits in Umformung berechnet:

$c = (4, 1, 0{,}9998)^T$ ✓

$RP_2x = c$:

$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & 4 \\ -1/4 & 0{,}5 & 0 & 1 \\ -0{,}49995 & 0 & 0 & 0{,}9998 \end{array}\right) \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 4 & 4 \\ -1/4 & 0{,}5 & 0 & 1 \\ 1 & 0 & 0 & -1{,}9998 \end{array}\right)$$

$$\Leftrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 4 & 4 \\ 0 & 1 & 0 & 1{,}0001 \\ 1 & 0 & 0 & -1{,}9998 \end{array}\right) \Leftrightarrow \left(\begin{array}{ccc|c} 0 & 0 & 1 & 0{,}9999 \\ 0 & 1 & 0 & 1{,}0001 \\ 1 & 0 & 0 & -1{,}9998 \end{array}\right)$$

$\Rightarrow x = (-1{,}9998, 1{,}0001, 0{,}9999)^T$ ✓